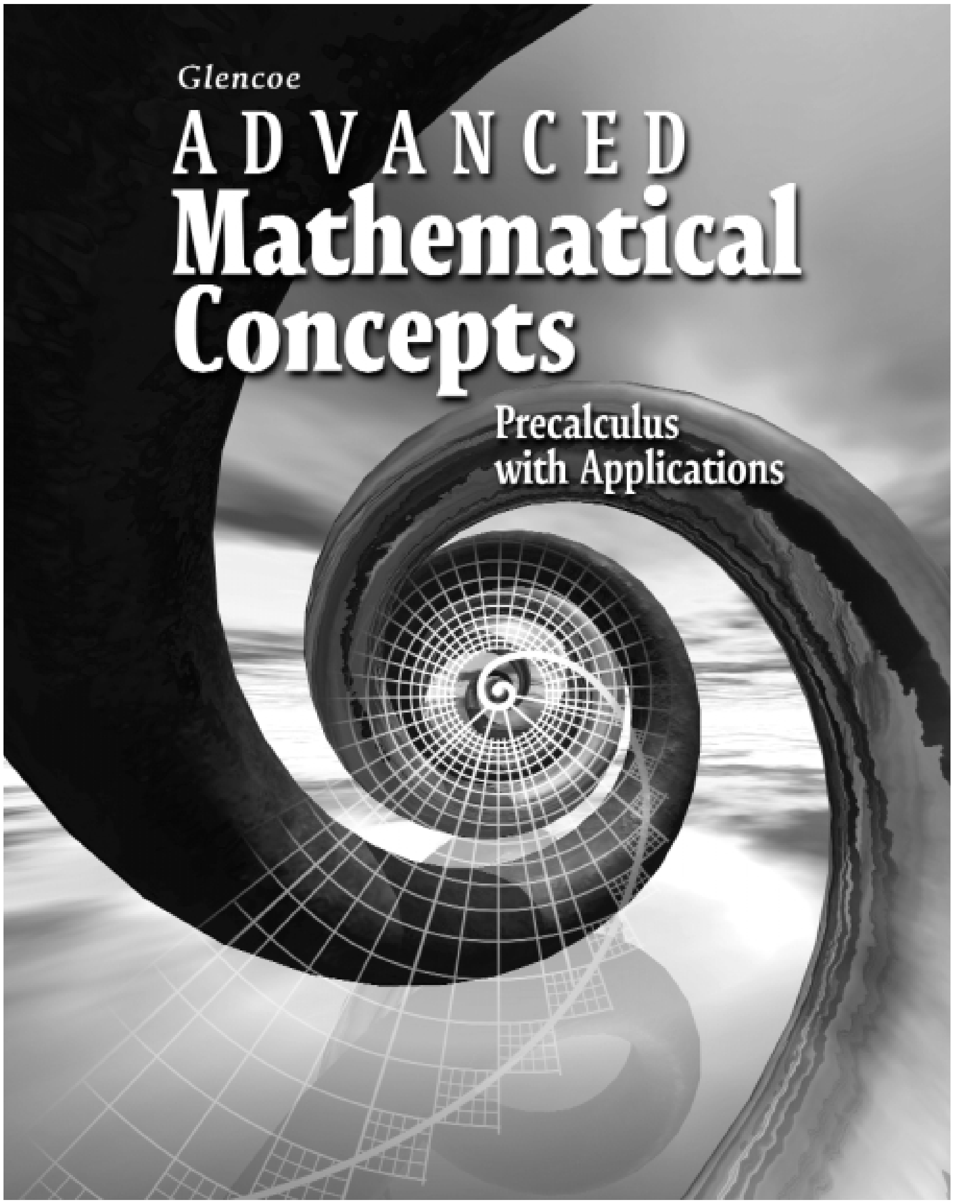
Study Guide



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Relations and Functions

A **relation** is a set of ordered pairs. The set of first elements in the ordered pairs is the **domain**, while the set of second elements is the **range**.

Example 1 **State the domain and range of the following relation.**
{(5, 2), (30, 8), (15, 3), (17, 6), (14, 9)}
Domain: {5, 14, 15, 17, 30}
Range: {2, 3, 6, 8, 9}
You can also use a table, a graph, or a rule to represent a relation.

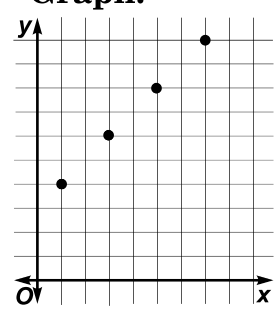
Example 2 **The domain of a relation is all odd positive integers less than 9. The range y of the relation is 3 more than x, where x is a member of the domain. Write the relation as a table of values and as an equation. Then graph the relation.**

Table:

x	y
1	4
3	6
5	8
7	10

Equation:
$y = x + 3$

Graph:



A **function** is a relation in which each element of the domain is paired with exactly one element in the range.

Example 3 **State the domain and range of each relation. Then state whether the relation is a function.**

a. {(−2, 1), (3, −1), (2, 0)}
The domain is {−2, 2, 3} and the range is {−1, 0, 1}. Each element of the domain is paired with exactly one element of the range, so this relation is a function.

b. {(3, −1), (3, −2), (9, 1)}
The domain is {3, 9}, and the range is {−2, −1, 1}. In the domain, 3 is paired with two elements in the range, −1 and −2. Therefore, this relation is not a function.

Example 4 **Evaluate each function for the given value.**

a. $f(-1)$ if $f(x) = 2x^3 + 4x^2 - 5x$

$$f(-1) = 2(-1)^3 + 4(-1)^2 - 5(-1) \quad x = -1$$
$$= -2 + 4 + 5 \text{ or } 7$$

b. $g(4)$ if $g(x) = x^4 - 3x^2 + 4$

$$g(4) = (4)^4 - 3(4)^2 + 4 \quad x = 4$$
$$= 256 - 48 + 4 \text{ or } 212$$

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Composition of Functions

Operations of Functions	Two functions can be added together, subtracted, multiplied, or divided to form a new function.

Example 1 **Given $f(x) = x^2 - x - 6$ and $g(x) = x + 2$, find each function.**

a. $(f + g)(x)$

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ &= x^2 - x - 6 + x + 2\\ &= x^2 - 4\end{aligned}$$

b. $(f - g)(x)$

$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= x^2 - x - 6 - (x + 2)\\ &= x^2 - 2x - 8\end{aligned}$$

c. $(f \cdot g)(x)$

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x)\\ &= (x^2 - x - 6)(x + 2)\\ &= x^3 + x^2 - 8x - 12\end{aligned}$$

d. $\left(\frac{f}{g}\right)(x)$

$$\begin{aligned}\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}\\ &= \frac{x^2 - x - 6}{x + 2}\\ &= \frac{(x - 3)(x + 2)}{x + 2}\\ &= x - 3,\ x \neq -2\end{aligned}$$

Functions can also be combined by using **composition**. The function formed by composing two functions f and g is called the **composite** of f and g, and is denoted by $f \circ g$. $[f \circ g](x)$ is found by substituting $g(x)$ for x in $f(x)$.

Example 2 **Given $f(x) = 3x^2 + 2x - 1$ and $g(x) = 4x + 2$, find $[f \circ g](x)$ and $[g \circ f](x)$.**

$$\begin{aligned}[f \circ g](x) &= f(g(x))\\ &= f(4x + 2) && \textit{Substitute } 4x + 2 \textit{ for } g(x).\\ &= 3(4x + 2)^2 + 2(4x + 2) - 1 && \textit{Substitute } 4x + 2 \textit{ for } x \textit{ in } f(x).\\ &= 3(16x^2 + 16x + 4) + 8x + 4 - 1\\ &= 48x^2 + 56x + 15\end{aligned}$$

$$\begin{aligned}[g \circ f](x) &= g(f(x))\\ &= g(3x^2 + 2x - 1) && \textit{Substitute } 3x^2 + 2x - 1 \textit{ for } f(x).\\ &= 4(3x^2 + 2x - 1) + 2 && \textit{Substitute } 3x^2 + 2x - 1 \textit{ for } x \textit{ in } g(x).\\ &= 12x^2 + 8x - 2\end{aligned}$$

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Graphing Linear Equations

You can graph a **linear equation** $Ax + By + C = 0$, where A and B are not both zero, by using the x- and y-intercepts. To find the x-intercept, let $y = 0$. To find the y-intercept, let $x = 0$.

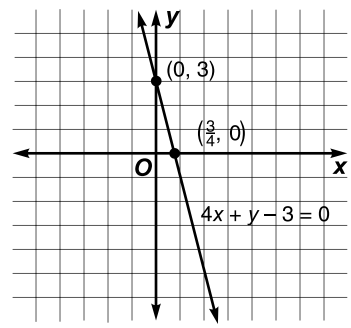
Example 1 **Graph $4x + y - 3 = 0$ using the x- and y-intercepts.**
Substitute 0 for y to find the x-intercept. Then substitute 0 for x to find the y-intercept.

x-intercept

$$\begin{aligned} 4x + y - 3 &= 0 \\ 4x + 0 - 3 &= 0 \\ 4x - 3 &= 0 \\ 4x &= 3 \\ x &= \frac{3}{4} \end{aligned}$$

y-intercept

$$\begin{aligned} 4x + y - 3 &= 0 \\ 4(0) + y - 3 &= 0 \\ y - 3 &= 0 \\ y &= 3 \end{aligned}$$



The line crosses the x-axis at $\left(\frac{3}{4}, 0\right)$ and the y-axis at $(0, 3)$.

Graph the intercepts and draw the line that passes through them.

The **slope** of a nonvertical line is the ratio of the change in the y-coordinates of two points to the corresponding change in the x-coordinates of the same points. The slope of a line can be interpreted as the ratio of change in the y-coordinates to the change in the x-coordinates.

Slope	The slope m of a line through two points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Example 2 **Find the slope of the line passing through $A(-3, 5)$ and $B(6, 2)$.**

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 5}{6 - (-3)} \qquad \textit{Let } x_1 = -3,\ y_1 = 5,\ x_2 = 6, \textit{ and } y_2 = 2. \\ &= \frac{-3}{9} \text{ or } -\frac{1}{3} \end{aligned}$$

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Writing Linear Equations

The form in which you write an equation of a line depends on the information you are given. Given the slope and y-intercept, or given the slope and one point on the line, the **slope-intercept** form can be used to write the equation.

Example 1 Write an equation in slope-intercept form for each line described.

a. a slope of $\frac{2}{3}$ and a y-intercept of -5

Substitute $\frac{2}{3}$ for m and -5 for b in the general slope-intercept form.

$y = mx + b \quad \rightarrow \quad y = \frac{2}{3}x - 5.$

The slope-intercept form of the equation of the line is $y = \frac{2}{3}x - 5$.

b. a slope of 4 and passes through the point at $(-2, 3)$

Substitute the slope and coordinates of the point in the general slope-intercept form of a linear equation. Then solve for b.

$y = mx + b$

$3 = 4(-2) + b$ *Substitute -2 for x, 3 for y, and 4 for m.*

$11 = b$ *Add 8 to both sides of the equation.*

The y-intercept is 11. Thus, the equation for the line is $y = 4x + 11$.

When you know the coordinates of two points on a line, you can find the slope of the line. Then the equation of the line can be written using either the slope-intercept or the **point-slope** form, which is $y - y_1 = m(x - x_1)$.

Example 2 ***Sales*** **In 1998, the average weekly first-quarter sales at Vic's Hardware store were \$9250. In 1999, the average weekly first-quarter sales were \$10,100. Assuming a linear relationship, find the average quarterly rate of increase.**

(1, 9250) and (5, 10,100) *Since there are two data points, identify the two coordinates to find the slope of the line. Coordinate 1 represents the first quarter of 1998 and coordinate 5 represents the first quarter of 1999.*

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{10{,}100 - 9250}{5 - 1}$$

$$= \frac{850}{4} \text{ or } 212.5$$

Thus, for each quarter, the average sales increase was \$212.50.

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Writing Equations of Parallel and Perpendicular Lines

- Two nonvertical lines in a plane are **parallel** if and only if their slopes are equal and they have no points in common.
- Graphs of two equations that represent the same line are said to **coincide**.
- Two nonvertical lines in a plane are **perpendicular** if and only if their slopes are negative reciprocals.

Example 1 **Determine whether the graphs of each pair of equations are *parallel*, *coinciding*, or *neither*.**

a. $2x - 3y = 5$, $6x - 9y = 21$

b. $12x + 6y = 18$, $4x = -2y + 6$

Write each pair of equations in slope-intercept form.

a. $2x - 3y = 5$ $\quad$ $6x - 9y = 21$

$y = \frac{2}{3}x - \frac{5}{3}$ $\quad$ $y = \frac{2}{3}x - \frac{7}{3}$

The lines have the same slope but different y-intercepts, so they are parallel.

b. $12x + 6y = 18$ $\quad$ $4x = -2y + 6$

$y = -2x + 3$ $\quad$ $y = -2x + 3$

The equations are identical, so the lines coincide.

Example 2 **Write the standard form of the equation of the line that passes through the point at (3, −4) and is parallel to the graph of $x + 3y - 4 = 0$.**

Any line parallel to the graph of $x + 3y - 4 = 0$ will have the same slope. So, find the slope of the graph of $x + 3y - 4 = 0$.

$$m = -\frac{A}{B}$$

$$= -\frac{1}{3} \quad A = 1, B = 3$$

Use the point-slope form to write the equation of the line.

$$y - y_1 = m(x - x_1)$$

$$y - (-4) = -\frac{1}{3}(x - 3) \quad x_1 = 3, y_1 = -4, m = -\frac{1}{3}$$

$$y + 4 = -\frac{1}{3}x + 1$$

$$3y + 12 = -x + 3 \quad \textit{Multiply each side by 3.}$$

$$x + 3y + 9 = 0 \quad \textit{Write in standard form.}$$

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Modeling Real-World Data with Linear Functions

When real-world data are collected, the data graphed usually do not form a straight line. However, the graph may approximate a linear relationship.

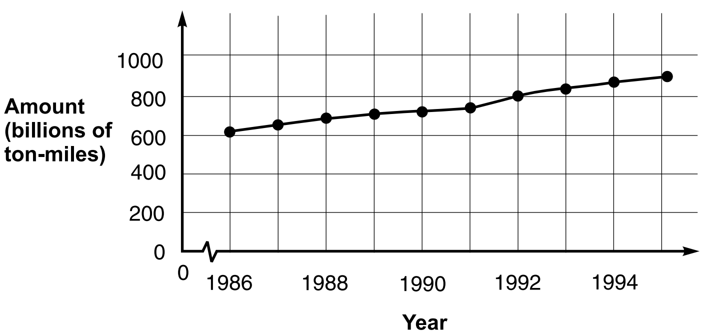
Example **The table shows the amount of freight hauled by trucks in the United States. Use the data to draw a line of best fit and to predict the amount of freight that will be carried by trucks in the year 2010.**

U.S. Truck Freight Traffic

Year	Amount (billions of ton-miles)
1986	632
1987	663
1988	700
1989	716
1990	735
1991	758
1992	815
1993	861
1994	908
1995	921

Source: *Transportation in America*

Graph the data on a scatter plot. Use the year as the independent variable and the ton-miles as the dependent variable. Draw a line of best fit, with some points on the line and others close to it.



Write a prediction equation for the data. Select two points that appear to represent the data. We chose (1990, 735) and (1993, 861).

Determine the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \rightarrow \frac{861 - 735}{1993 - 1990} = \frac{126}{3} \text{ or } 42$$

Use one of the ordered pairs, such as (1990, 735), and the slope in the point-slope form of the equation.

$$y - y_1 = m(x - x_1)$$
$$y - 735 = 42(x - 1990)$$
$$y = 42x - 82{,}845$$

A prediction equation is $y = 42x - 82{,}845$. Substitute 2010 for x to estimate the average amount of freight a truck will haul in 2010.

$$y = 42x - 82{,}845$$
$$y = 42(2010) - 82{,}845$$
$$y = 1575$$

According to this prediction equation, trucks will haul 1575 billion ton-miles in 2010.

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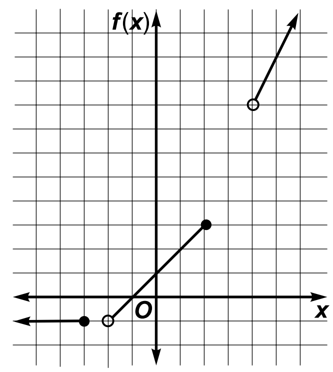
Piecewise Functions

Piecewise functions use different equations for different intervals of the domain. When graphing piecewise functions, the partial graphs over various intervals do not necessarily connect.

Example 1 **Graph** $f(x) = \begin{cases} -1 & \text{if } x \leq -3 \\ 1 + x & \text{if } -2 < x \leq 2 \\ 2x & \text{if } x > 4 \end{cases}$



First, graph the constant function $f(x) = -1$ for $x \leq -3$. This graph is a horizontal line. Because the point at $(-3, -1)$ is included in the graph, draw a closed circle at that point.

Second, graph the function $f(x) = 1 + x$ for $-2 < x \leq 2$. Because $x = -2$ is not included in this region of the domain, draw an open circle at $(-2, -1)$. The value of $x = 2$ is included in the domain, so draw a closed circle at $(2, 3)$ since for $f(x) = 1 + x$, $f(2) = 3$.

Third, graph the line $f(x) = 2x$ for $x > 4$. Draw an open circle at $(4, 8)$ since for $f(x) = 2x$, $f(4) = 8$.

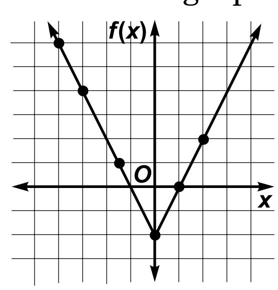
A piecewise function whose graph looks like a set of stairs is called a **step function**. One type of step function is the **greatest integer function**. The symbol $[\![x]\!]$ means *the greatest integer not greater than x*. The graphs of step functions are often used to model real-world problems such as fees for phone services and the cost of shipping an item of a given weight.

The **absolute value function** is another piecewise function. Consider $f(x) = |x|$. The absolute value of a number is always nonnegative.

Example 2 **Graph** $f(x) = 2|x| - 2$.

Use a table of values to determine points on the graph.

x	$2\|x\| - 2$	$(x, f(x))$
-4	$2\|-4\| - 2$	$(-4, 6)$
-3	$2\|-3\| - 2$	$(-3, 4)$
-1.5	$2\|-1.5\| - 2$	$(-1.5, 1)$
0	$2\|0\| - 2$	$(0, -2)$
1	$2\|1\| - 2$	$(1, 0)$
2	$2\|2\| - 2$	$(2, 2)$



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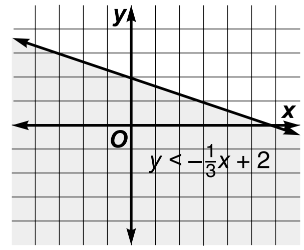
Graphing Linear Inequalities



The graph of $y = -\frac{1}{3}x + 2$ is a line that separates the coordinate plane into two regions, called **half planes**. The line described by $y = -\frac{1}{3}x + 2$ is called the **boundary** of each region. If the boundary is part of a graph, it is drawn as a solid line. A boundary that is not part of the graph is drawn as a dashed line.

The graph of $y > -\frac{1}{3}x + 2$ is the region above the line. The graph of $y < -\frac{1}{3}x + 2$ is the region below the line.

You can determine which half plane to shade by testing a point on either side of the boundary in the original inequality. If it is not on the boundary, (0, 0) is often an easy point to test. If the inequality is true for your test point, then shade the half plane that contains the test point. If the inequality is false for your test point, then shade the half plane that does not contain the test point.

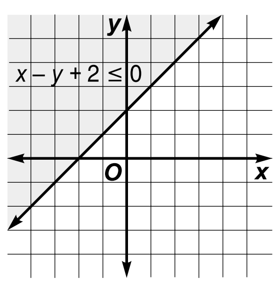
Example **Graph each inequality.**

a. $\boldsymbol{x - y + 2 \leq 0}$

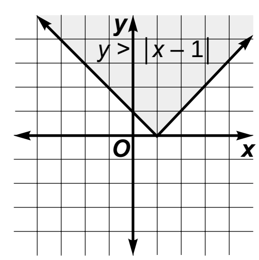
$$\begin{aligned} x - y + 2 &\leq 0 \\ -y &\leq -x - 2 \\ y &\geq x + 2 \end{aligned}$$

Reverse the inequality when you divide or multiply by a negative.

The graph does include the boundary, so the line is solid. Testing (0, 0) in the inequality yields a false inequality, $0 \geq 2$. Shade the half plane that does not include (0, 0).



b. $\boldsymbol{y > |x - 1|}$



Graph the equation with a dashed boundary. Then test a point to determine which region is shaded. The test point (0, 0) yields the false inequality $0 > 1$, so shade the region that does not include (0, 0).

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Study Guide

Solving Systems of Equations in Two Variables

One way to solve a system of equations in two variables is by graphing. The intersection of the graphs is called the solution to the **system of equations.**

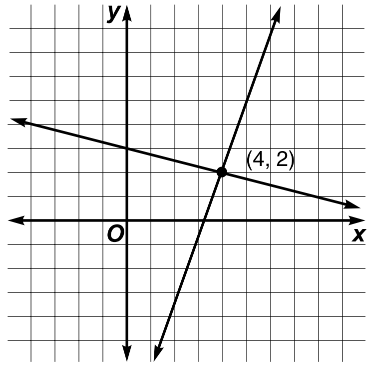
Example 1 **Solve the system of equations by graphing.**
$\boldsymbol{3x - y = 10}$
$\boldsymbol{x + 4y = 12}$

First rewrite each equation of the system in slope-intercept form by solving for y.

$3x - y = 10 \quad \xrightarrow{\text{becomes}} \quad y = 3x - 10$

$x + 4y = 12 \quad \xrightarrow{\text{becomes}} \quad y = -\frac{1}{4}x + 3$

The solution to the system is (4, 2).



A **consistent** system has at least one solution.
A system having exactly one solution is **independent.**
If a system has infinitely many solutions, the system is **dependent.**
Systems that have no solution are **inconsistent.**

Systems of linear equations can also be solved algebraically using the **elimination method** or the **substitution method.**

Example 2
Use the elimination method to solve the system of equations.
$\boldsymbol{2x - 3y = -21}$
$\boldsymbol{5x + 6y = 15}$

To solve this system, multiply each side of the first equation by 2, and add the two equations to eliminate y. Then solve the resulting equation.

$2(2x - 3y) = 2(-21) \rightarrow 4x - 6y = -42$

$$\begin{array}{rl} 4x - 6y = & -42 \\ 5x + 6y = & 15 \\ \hline 9x \qquad = & -27 \\ x = & -3 \end{array}$$

Now substitute -3 for x in either of the *original* equations.

$$\begin{aligned} 5x + 6y &= 15 \\ 5(-3) + 6y &= 15 \qquad x = -3 \\ 6y &= 30 \\ y &= 5 \end{aligned}$$

The solution is $(-3, 5)$.

Example 3
Use the substitution method to solve the system of equations.
$\boldsymbol{x = 7y + 3}$
$\boldsymbol{2x - y = -7}$

The first equation is stated in terms of x, so substitute $7y + 3$ for x in the second equation.

$$\begin{aligned} 2x - y &= -7 \\ 2(7y + 3) - y &= -7 \\ 13y &= -13 \\ y &= -1 \end{aligned}$$

Now solve for x by substituting -1 for y in either of the *original* equations.

$x = 7y + 3$
$x = 7(-1) + 3 \qquad y = -1$
$x = -4$

The solution is $(-4, -1)$.

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Study Guide

Solving Systems of Equations in Three Variables

You can solve systems of three equations in three variables using the same algebraic techniques you used to solve systems of two equations.

Example **Solve the system of equations by elimination.**

$$\begin{aligned}4x + y + 2z &= -8\\ 2x - 3y - 4z &= 4\\ -7x + 2y + 3z &= 2\end{aligned}$$

Choose a pair of equations and then eliminate one of the variables. Because the coefficient of y is 1 in the first equation, eliminate y from the second and third equations.

To eliminate y using the first and second equations, multiply both sides of the first equation by 3.

$$3(4x + y + 2z) = 3(-8)$$
$$12x + 3y + 6z = -24$$

Then add the second equation to that result.

$$\begin{array}{rcrcrcr} 12x & + & 3y & + & 6z & = & -24\\ 2x & - & 3y & - & 4z & = & 4\\ \hline 14x & & & + & 2z & = & -20\end{array}$$

To eliminate y using the first and third equations, multiply both sides of the first equation by -2.

$$-2(4x + y + 2z) = -2(-8)$$
$$-8x - 2y - 4z = 16$$

Then add the second equation to that result.

$$\begin{array}{rcrcrcr} -8x & - & 2y & - & 4z & = & 16\\ -7x & + & 2y & + & 3z & = & 2\\ \hline -15x & & & & -z & = & 18\end{array}$$

Now you have two linear equations in two variables. Solve this system. Eliminate z by multiplying both sides of the second equation by 2. Then add the two equations.

$$2(-15x - z) = 2(18)$$
$$-30x - 2z = 36$$

$$\begin{array}{rcrcr} 14x & + & 2z & = & -20\\ -30x & - & 2z & = & 36\\ \hline -16x & & & = & 16\\ & & x & = & -1\end{array}$$

The value of x is -1.

By substituting the value of x into one of the equations in two variables, we can solve for the value of z.

$$14x + 2z = -20$$
$$14(-1) + 2z = -20 \qquad x = -1$$
$$z = -3$$

The value of z is -3.

Finally, use one of the original equations to find the value of y.

$$4x + y + 2z = -8$$
$$4(-1) + y + 2(-3) = -8 \qquad x = -1, z = -3$$
$$y = 2$$

The value of y is 2.

The solution is $x = -1$, $y = 2$, $z = -3$. This can be written as the **ordered triple** $(-1, 2, -3)$.

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Modeling Real-World Data with Matrices

A **matrix** is a rectangular array of terms called **elements.**
A matrix with m rows and n columns is an $\boldsymbol{m \times n}$ **matrix.**
The dimensions of the matrix are m and n.

Example 1 **Find the values of x and y for which** $\begin{bmatrix} 3x + 5 \\ y \end{bmatrix} = \begin{bmatrix} y - 9 \\ -4x \end{bmatrix}$ **is true.**

Since the corresponding elements must be equal, we can express the equal matrices as two equations.

$3x + 5 = y - 9$
$y = -4x$

Solve the system of equations by using substitution.

$3x + 5 = y - 9$
$3x + 5 = (-4x) - 9$ *Substitute $-4x$ for y.*
$x = -2$

$y = -4x$
$y = -4(-2)$ *Substitute -2 for x.*
$y = 8$

The matrices are equal if $x = -2$ and $y = 8$.

To add or subtract matrices, the dimensions of the matrices must be the same.

Example 2 **Find $C - D$ if $C = \begin{bmatrix} 3 & 6 \\ -2 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 5 \\ -7 & 8 \end{bmatrix}$.**

$$\begin{aligned} C - D &= C + (-D) \\ &= \begin{bmatrix} 3 & 6 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -5 \\ 7 & -8 \end{bmatrix} \\ &= \begin{bmatrix} 3 + (-1) & 6 + (-5) \\ -2 + 7 & 4 + (-8) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 5 & -4 \end{bmatrix} \end{aligned}$$

To multiply two matrices, the number of columns in the first matrix must be equal to the number of rows in the second matrix. The product of two matrices is found by multiplying columns and rows.

Example 3 **Find each product if $Z = \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix}$ and $Y = \begin{bmatrix} 8 & 0 \\ 9 & -6 \end{bmatrix}$.**

a. XY

$$XY = \begin{bmatrix} 4(8) + 3(9) & 4(0) + 3(-6) \\ 5(8) + 2(9) & 5(0) + 2(-6) \end{bmatrix} \text{ or } \begin{bmatrix} 59 & -18 \\ 58 & -12 \end{bmatrix}$$

b. YX

$$YX = \begin{bmatrix} 8(4) + 0(5) & 8(3) + 0(2) \\ 9(4) + (-6)(5) & 9(3) + (-6)(2) \end{bmatrix} \text{ or } \begin{bmatrix} 32 & 24 \\ 6 & 15 \end{bmatrix}$$

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Modeling Motion with Matrices

You can use matrices to perform many **transformations,** such as **translations** (slides), **reflections** (flips), **rotations** (turns), and **dilations** (enlargements or reductions).

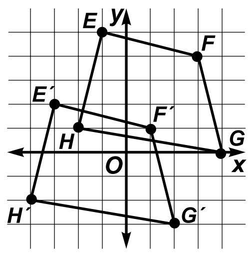
Example 1 **Suppose quadrilateral *EFGH* with vertices *E*(−1, 5), *F*(3, 4), *G*(4, 0), and *H*(−2, 1) is translated 2 units left and 3 units down.**

The **vertex matrix** for the quadrilateral is $\begin{bmatrix} -1 & 3 & 4 & -2 \\ 5 & 4 & 0 & 1 \end{bmatrix}$.

The **translation matrix** is $\begin{bmatrix} -2 & -2 & -2 & -2 \\ -3 & -3 & -3 & -3 \end{bmatrix}$.

Adding the two matrices gives the coordinates of the vertices of the translated quadrilateral.

$$\begin{bmatrix} -1 & 3 & 4 & -2 \\ 5 & 4 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -2 & -2 & -2 \\ -3 & -3 & -3 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 2 & -4 \\ 2 & 1 & -3 & -2 \end{bmatrix}$$



Graphing the **pre-image** and the **image** of the translated quadrilateral on the same axes, we can see the effect of the translation.

The chart below summarizes the matrices needed to produce specific reflections or rotations. All rotations are counterclockwise about the origin.

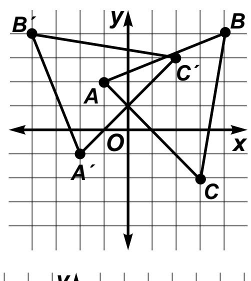
Reflections	$R_{x\text{-axis}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$R_{y\text{-axis}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$R_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Rotations	$Rot_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$Rot_{180} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$Rot_{270} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Example 2 **A triangle has vertices *A*(−1, 2), *B*(4, 4), and *C*(3, −2). Find the coordinates of the image of the triangle after a rotation of 90° counterclockwise about the origin.**

The vertex matrix is $\begin{bmatrix} -1 & 4 & 3 \\ 2 & 4 & -2 \end{bmatrix}$.

Multiply it by the 90° **rotation matrix.**

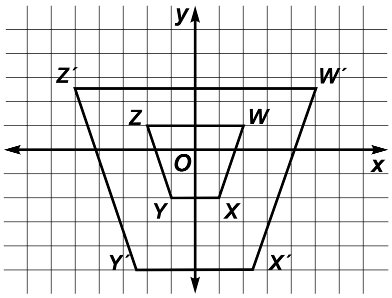
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 & 3 \\ 2 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 2 \\ -1 & 4 & 3 \end{bmatrix}$$



Example 3 **Trapezoid *WXYZ* has vertices *W*(2, 1), *X*(1, −2), *Y*(−1, −2), and *Z*(−2, 1). Find the coordinates of dilated trapezoid *W′X′Y′Z′* for a scale factor of 2.5.**

Perform scalar multiplication on the vertex matrix for the trapezoid.

$$2.5\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & -2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2.5 & -2.5 & -5 \\ 2.5 & -5 & -5 & 2.5 \end{bmatrix}$$



2-5

Study Guide

Determinants and Multiplicative Inverses of Matrices

Each square matrix has a **determinant.** The determinant of a 2×2 matrix is a number denoted by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ or $\det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$. Its value is $a_1b_2 - a_2b_1$.

Example 1 **Find the value of** $\begin{vmatrix} \mathbf{5} & \mathbf{-2} \\ \mathbf{3} & \mathbf{1} \end{vmatrix}$.

$$\begin{vmatrix} 5 & -2 \\ 3 & 1 \end{vmatrix} = 5(1) - 3(-2) \text{ or } 11$$

The **minor** of an element can be found by deleting the row and column containing the element.

$$\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix}$$

The minor of a_1 *is* $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$.

Example 2 **Find the value of** $\begin{vmatrix} \mathbf{8} & \mathbf{9} & \mathbf{3} \\ \mathbf{3} & \mathbf{5} & \mathbf{7} \\ \mathbf{-1} & \mathbf{2} & \mathbf{4} \end{vmatrix}$.

$$\begin{vmatrix} 8 & 9 & 3 \\ 3 & 5 & 7 \\ -1 & 2 & 4 \end{vmatrix} = 8\begin{vmatrix} 5 & 7 \\ 2 & 4 \end{vmatrix} - 9\begin{vmatrix} 3 & 7 \\ -1 & 4 \end{vmatrix} + 3\begin{vmatrix} 3 & 5 \\ -1 & 2 \end{vmatrix}$$
$$= 8(6) - 9(19) + 3(11) \text{ or } -90$$

The multiplicative inverse of a matrix is defined as follows.

$$\text{If } A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \text{ and } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \text{ then } A^{-1} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}.$$

Example 3 **Solve the system of equations by using matrix equations.**
$\mathbf{5x + 4y = -3}$
$\mathbf{3x - 5y = -24}$

Write the system as a matrix equation.

$$\begin{bmatrix} 5 & 4 \\ 3 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -24 \end{bmatrix}$$

To solve the matrix equation, first find the inverse of the coefficient matrix.

$$\frac{1}{\begin{vmatrix} 5 & 4 \\ 3 & -5 \end{vmatrix}}\begin{bmatrix} -5 & -4 \\ -3 & 5 \end{bmatrix} = \frac{1}{37}\begin{bmatrix} -5 & -4 \\ -3 & 5 \end{bmatrix}$$

Now multiply each side of the matrix equation by the inverse and solve.

$$-\frac{1}{37}\begin{bmatrix} -5 & -4 \\ -3 & 5 \end{bmatrix}.$$

$$\begin{bmatrix} 5 & 4 \\ 3 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{37}\begin{bmatrix} -5 & -4 \\ -3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -24 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

The solution is $(-3, 3)$.

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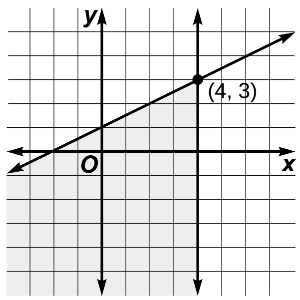
Study Guide

Solving Systems of Linear Inequalities

To solve a **system of linear inequalities**, you must find the ordered pairs that satisfy all inequalities. One way to do this is to graph the inequalities on the same coordinate plane. The intersection of the graphs contains points with ordered pairs in the solution set. If the graphs of the inequalities do not intersect, then the system has no solution.

Example 1 **Solve the system of inequalities by graphing.** $-x + 2y \leq 2$, $x \leq 4$

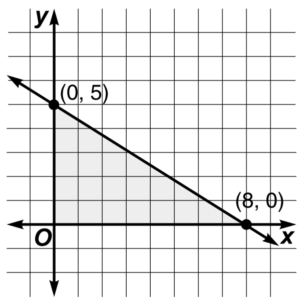
The shaded region represents the solution to the system of inequalities.



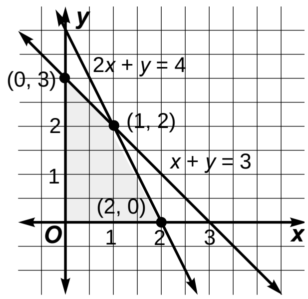
A system of more than two linear inequalities can have a solution that is a bounded set of points called a **polygonal convex set.**

Example 2 **Solve the system of inequalities by graphing and name the coordinates of the vertices of the polygonal convex set.** $x \geq 0$, $y \geq 0$, $5x + 8y \leq 40$

The region shows points that satisfy all three inequalities. The region is a triangle whose vertices are the points at (0, 0), (0, 5), and (8, 0).



Example 3 **Find the maximum and minimum values of $f(x, y) = x + 2y + 1$ for the polygonal convex set determined by the following inequalities.**
$x \geq 0 \quad y \geq 0 \quad 2x + y \leq 4 \quad x + y \leq 3$



First, graph the inequalities and find the coordinates of the vertices of the resulting polygon.

The coordinates of the vertices are (0, 0), (2, 0), (1, 2), and (0, 3).

Then, evaluate the function $f(x, y) = x + 2y + 1$ at each vertex.

$f(0, 0) = 0 + 2(0) + 1 = 1$ $\qquad$ $f(1, 2) = 1 + 2(2) + 1 = 6$

$f(2, 0) = 2 + 2(0) + 1 = 3$ $\qquad$ $f(0, 3) = 0 + 2(3) + 1 = 7$

Thus, the maximum value of the function is 7, and the minimum value is 1.

2-7

Study Guide

Linear Programming

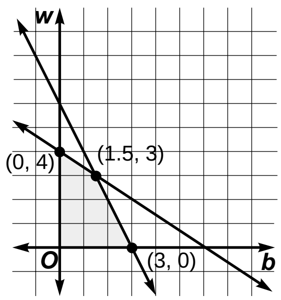
The following example outlines the procedure used to solve **linear programming** problems.

Example **The B & W Leather Company wants to add handmade belts and wallets to its product line. Each belt nets the company \$18 in profit, and each wallet nets \$12. Both belts and wallets require cutting and sewing. Belts require 2 hours of cutting time and 6 hours of sewing time. Wallets require 3 hours of cutting time and 3 hours of sewing time. If the cutting machine is available 12 hours a week and the sewing machine is available 18 hours per week, what mix of belts and wallets will produce the most profit within the constraints?**

Define variables. Let b = the number of belts.
Let w = the number of wallets.

Write inequalities.
$b \geq 0$
$w \geq 0$
$2b + 3w \leq 12$ cutting
$6b + 3w \leq 18$ sewing

Graph the system.



Write an equation. Since the profit on belts is \$18 and the profit on wallets is \$12, the profit function is $B(b, w) = 18b + 12w$.

Substitute values.
$B(0, 0) = 18(0) + 12\,(0) = 0$
$B(0, 4) = 18(0) + 12(4) = 48$
$B(1.5, 3) = 18(1.5) + 12(3) = 63$
$B(3, 0) = 18(3) + 12(0) = 54$

Answer the problem. The B & W Company will maximize profit if it makes and sells 1.5 belts for every 3 wallets.

When constraints of a linear programming problem cannot be satisfied simultaneously, then **infeasibility** is said to occur.

The solution of a linear programming problem is **unbounded** if the region defined by the constraints is infinitely large.

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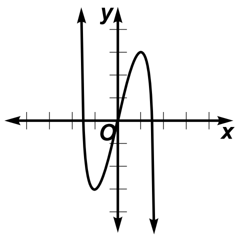
3-1

Study Guide

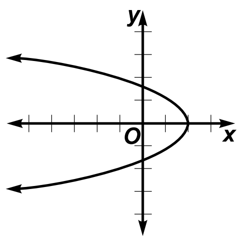
Symmetry and Coordinate Graphs

One type of symmetry a graph may have is **point symmetry.** A common point of symmetry is the origin. Another type is **line symmetry.** Some common lines of symmetry are the x-axis, the y-axis, and the lines $y = x$ and $y = -x$.

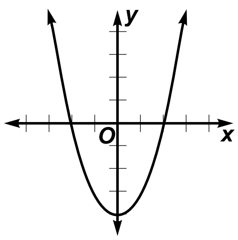
point



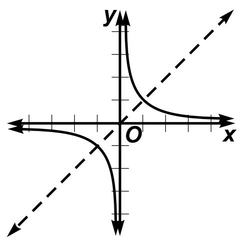
x-axis



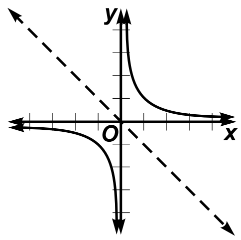
y-axis



$y = x$



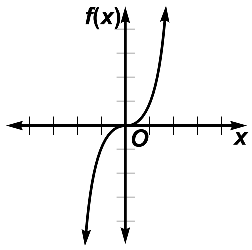
$y = -x$



Example 1 **Determine whether $f(x) = x^3$ is symmetric with respect to the origin.**

If $f(-x) = -f(x)$, the graph has point symmetry.

Find $f(-x)$.	Find $-f(x)$.
$f(-x) = (-x)^3$	$-f(x) = -x^3$
$f(-x) = -x^3$	



The graph of $f(x) = x^3$ is symmetric with respect to the origin because $f(-x) = -f(x)$.

Example 2 **Determine whether the graph of $x^2 + 2 = y^2$ is symmetric with respect to the x-axis, the y-axis, the line $y = x$, the line $y = -x$, or none of these.**

Substituting (a, b) into the equation yields $a^2 + 2 = b^2$. Check to see if each test produces an equation equivalent to $a^2 + 2 = b^2$.

x-axis	$a^2 + 2 = (-b)^2$	*Substitute $(a, -b)$ into the equation.*
	$a^2 + 2 = b^2$	*Equivalent to $a^2 + 2 = b^2$*
y-axis	$(-a)^2 + 2 = b^2$	*Substitute $(-a, b)$ into the equation.*
	$a^2 + 2 = b^2$	*Equivalent to $a^2 + 2 = b^2$*
$y = x$	$(b)^2 + 2 = (a)^2$	*Substitute (b, a) into the equation.*
	$a^2 - 2 = b^2$	*Not equivalent to $a^2 + 2 = b^2$*
$y = -x$	$(-b)^2 + 2 = (-a)^2$	*Substitute $(-b, -a)$ into the equation.*
	$b^2 + 2 = a^2$	*Simplify.*
	$a^2 - 2 = b^2$	*Not equivalent to $a^2 + 2 = b^2$*

Therefore, the graph of $x^2 + 2 = y^2$ is symmetric with respect to the x-axis and the y-axis.

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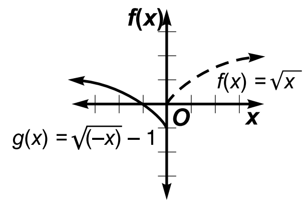
Study Guide

Families of Graphs

A **parent graph** is a basic graph that is transformed to create other members in a family of graphs. The transformed graph may appear in a different location, but it will resemble the parent graph.

A **reflection** flips a graph over a line called the *axis of symmetry.*
A **translation** moves a graph vertically or horizontally.
A **dilation** expands or compresses a graph vertically or horizontally.

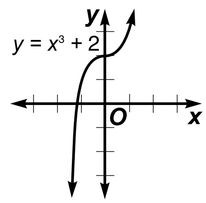
Example 1 **Describe how the graphs of $f(x) = \sqrt{x}$ and $g(x) = \sqrt{-x} - 1$ are related.**



The graph of $g(x)$ is a reflection of the graph of $f(x)$ over the y-axis and then translated down 1 unit.

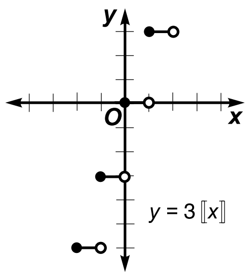
Example 2 **Use the graph of the given parent function to sketch the graph of each related function.**

a. $f(x) = x^3$; $y = x^3 + 2$



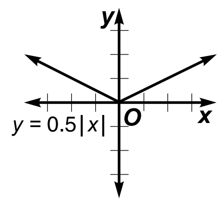
When 2 is added to the parent function, the graph of the parent function moves up 2 units.

b. $f(x) = [\![x]\!]$; $y = 3[\![x]\!]$



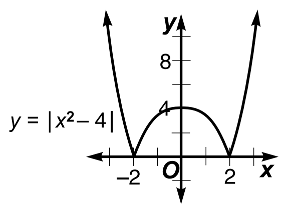
The parent function is expanded vertically by a factor of 3, so the vertical distance between the steps is 3 units.

c. $f(x) = |x|$; $y = 0.5|x|$



When $|x|$ is multiplied by a constant greater than 0 but less than 1, the graph compresses vertically, in this case, by a factor of 0.5.

d. $f(x) = x^2$; $y = |x^2 - 4|$



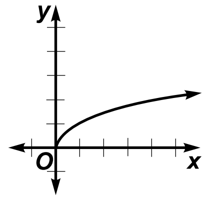
The parent function is translated down 4 units and then any portion of the graph below the x-axis is reflected so that it is above the x-axis.

Study Guide

Graphs of Nonlinear Inequalities

Graphing an inequality in two variables identifies all ordered pairs that satisfy the inequality. The first step in graphing nonlinear inequalities is graphing the boundary.

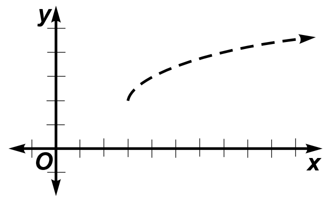
Example 1 **Graph $y < \sqrt{x-3} + 2$.**



The boundary of the inequality is the graph of $y = \sqrt{x-3} + 2$. To graph the boundary curve, start with the parent graph $y = \sqrt{x}$. Analyze the boundary equation to determine how the boundary relates to the parent graph.

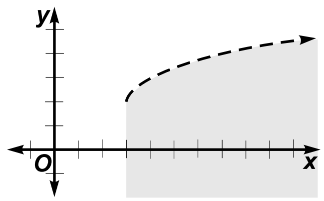
$$y = \sqrt{x - \underset{\uparrow}{3}} + \underset{\uparrow}{2}$$

move 3 units right *move 2 units up*



Since the boundary is not included in the inequality, the graph is drawn as a dashed curve.

The inequality states that the $y-$values of the solution are less than the $y-$values on the graph of $y = \sqrt{x-3} + 2$. Therefore, for a particular value of x, all of the points in the plane lie below the curve. This portion of the graph should be shaded.



To verify numerically, test a point not on the boundary.

$y < \sqrt{x-3} + 2$
$0 \overset{?}{<} \sqrt{4-3} + 2$ *Replace (x, y) with (4, 0).*
$0 < 3$ ✓ *True*

Since (4, 0) satisfies the inequality, the correct region is shaded.

Example 2 **Solve $|x - 3| - 2 > 7$.**

Two cases must be solved. In one case, $x - 3$ is negative, and in the other, $x - 3$ is positive.

Case 1 If $a < 0$, then $|a| = -a$.
$$-(x-3) - 2 > 7$$
$$-x + 3 - 2 > 7$$
$$-x > 6$$
$$x < -6$$

Case 2 If $a > 0$, then $|a| = a$.
$$x - 3 - 2 > 7$$
$$x - 5 > 7$$
$$x > 12$$

The solution set is $\{x \mid x < -6 \text{ or } x > 12\}$.

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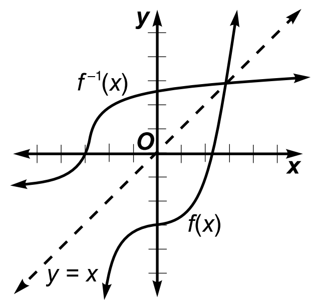
Study Guide

Inverse Functions and Relations

Two relations are inverse relations if and only if one relation contains the element (b, a) whenever the other relation contains the element (a, b). If $f(x)$ denotes a function, then $f^{-1}(x)$ denotes the inverse of $f(x)$.

Example 1 **Graph $f(x) = \frac{1}{4}x^3 - 3$ and its inverse.**

To graph the function, let $y = f(x)$. To graph $f^{-1}(x)$, interchange the x- and y-coordinates of the ordered pairs of the function.



$f(x) = \frac{1}{4}x^3 - 3$	
x	y
−3	−9.75
−2	−5
−1	−3.25
0	−3
1	−2.75
2	−1
3	3.75

$f^{-1}(x)$	
x	y
−9.75	−3
−5	−2
−3.25	−1
−3	0
−2.75	1
−1	2
3.75	3

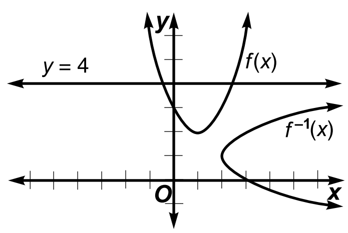
You can use the **horizontal line test** to determine if the inverse of a relation will be a function. If every horizontal line intersects the graph of the relation in at most one point, then the inverse of the relation is a function.

You can find the inverse of a relation algebraically. First, let $y = f(x)$. Then interchange x and y. Finally, solve the resulting equation for y.

Example 2 **Find the inverse of $f(x) = (x - 1)^2 + 2$. Determine if the inverse is a function.**

$y = (x - 1)^2 + 2$	*Let $y = f(x)$.*
$x = (y - 1)^2 + 2$	*Interchange x and y.*
$x - 2 = (y - 1)^2$	*Isolate the expression containing y.*
$\pm\sqrt{x - 2} = y - 1$	*Take the square root of each side.*
$y = 1 \pm \sqrt{x - 2}$	*Solve for y.*
$f^{-1}(x) = 1 \pm \sqrt{x - 2}$	*Replace y with $f^{-1}(x)$.*

Since the line $y = 4$ intersects the graph of $f(x)$ at more than one point, the function fails the horizontal line test. Thus, the inverse of $f(x)$ is not a function.



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Study Guide

Continuity and End Behavior

A function is **continuous** at $x = c$ if it satisfies the following three conditions.
(1) the function is defined at c; in other words, $f(c)$ exists;
(2) the function approaches the same y-value to the left and right of $x = c$; and
(3) the y-value that the function approaches from each side is $f(c)$.

Functions can be continuous or **discontinuous.** Graphs that are discontinuous can exhibit **infinite discontinuity**, **jump discontinuity**, or **point discontinuity.**

Example 1 **Determine whether each function is continuous at the given x-value. Justify your answer using the continuity test.**

a. $f(x) = 2|x| + 3; x = 2$

(1) The function is defined at $x = 2$; $f(2) = 7$.

(2) The tables below show that y approaches 7 as x approaches 2 from the left and that y approaches 7 as x approaches 2 from the right.

x	$y = f(x)$
1.9	6.8
1.99	6.98
1.999	6.998

x	$y = f(x)$
2.1	7.2
2.01	7.02
2.001	7.002

(3) Since the y-values approach 7 as x approaches 2 from both sides and $f(2) = 7$, the function is continuous at $x = 2$.

b. $f(x) = \frac{2x}{x^2 - 1}; x = 1$

Start with the first condition in the continuity test. The function is not defined at $x = 1$ because substituting 1 for x results in a denominator of zero. So the function is discontinuous at $x = 1$.

c. $f(x) = \begin{cases} 2x + 1 & \text{if } x > 2 \\ x - 1 & \text{if } x \le 2 \end{cases}; x = 2$

This function fails the second part of the continuity test because the values of $f(x)$ approach 1 as x approaches 2 from the left, but the values of $f(x)$ approach 5 as x approaches 2 from the right.

The **end behavior** of a function describes what the $y-$values do as $|x|$ becomes greater and greater. In general, the end behavior of any polynomial function can be modeled by the function made up solely of the term with the highest power of x and its coefficient.

Example 2 **Describe the end behavior of $p(x) = -x^5 + 2x^3 - 4$.**

Determine $f(x) = a_n x^n$ where x^n is the term in $p(x)$ with the highest power of x and a_n is its coefficient.

$f(x) = -x^5 \quad x^n = x^5 \quad a_n = -1$

Thus, by using the table on page 163 of your text, you can see that when a^n is negative and n is odd, the end behavior can be stated as $p(x) \to -\infty$ as $x \to \infty$ and $p(x) \to \infty$ as $x \to -\infty$.

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3-6

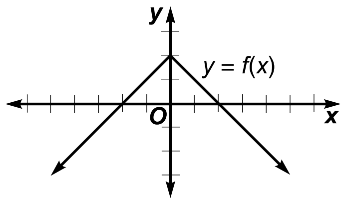
Study Guide

Critical Points and Extrema

Critical points are points on a graph at which a line drawn tangent to the curve is horizontal or vertical. A critical point may be a **maximum**, a **minimum**, or a **point of inflection.** A point of inflection is a point where the graph changes its curvature. Graphs can have an **absolute maximum**, an **absolute minimum**, a **relative maximum**, or a **relative minimum.** The general term for maximum or minimum is **extremum** (plural, *extrema*).

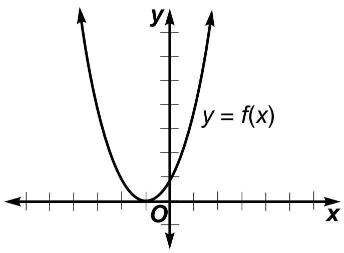
Example 1 **Locate the extrema for the graph of $y = f(x)$. Name and classify the extrema of the function.**

a.



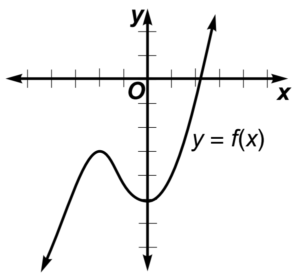
The function has an absolute maximum at (0, 2). The absolute maximum is the greatest value that a function assumes over its domain.

b.



The function has an absolute minimum at (−1, 0). The absolute minimum is the least value that a function assumes over its domain.

c.



The relative maximum and minimum may not be the greatest and the least y-value for the domain, respectively, but they are the greatest and least y-value on some interval of the domain. The function has a relative maximum at $(-2, -3)$ and a relative minimum at $(0, -5)$. Because the graph indicates that the function increases or decreases without bound as x increases or decreases, there is neither an absolute maximum nor an absolute minimum.

By testing points on both sides of a critical point, you can determine whether the critical point is a relative maximum, a relative minimum, or a point of inflection.

Example 2 **The function $f(x) = 2x^6 + 2x^4 - 9x^2$ has a critical point at $x = 0$. Determine whether the critical point is the location of a maximum, a minimum, or a point of inflection.**

x	$x - 0.1$	$x + 0.1$	$f(x - 0.1)$	$f(x)$	$f(x + 0.1)$	Type of Critical Point
0	−0.1	0.1	−0.0899	0	−0.0899	maximum

Because 0 is greater than both $f(x - 0.1)$ and $f(x + 0.1)$, $x = 0$ is the location of a relative maximum.

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Study Guide

Graphs of Rational Functions

A **rational function** is a quotient of two polynomial functions.

The line $x = a$ is a **vertical asymptote** for a function $f(x)$ if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$ from either the left or the right.

The line $y = b$ is a **horizontal asymptote** for a function $f(x)$ if $f(x) \to b$ as $x \to \infty$ or as $x \to -\infty$.

A **slant asymptote** occurs when the degree of the numerator of a rational function is exactly one greater than that of the denominator.

Example 1 **Determine the asymptotes for the graph of $f(x) = \frac{2x - 1}{x + 3}$.**

Since $f(-3)$ is undefined, there may be a vertical asymptote at $x = -3$. To verify that $x = -3$ is a vertical asymptote, check to see that $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to -3$ from either the left or the right.

x	$f(x)$
-2.9	-68
-2.99	-698
-2.999	-6998
-2.9999	-69998

The values in the table confirm that $f(x) \to -\infty$ as $x \to -3$ from the right, so there is a vertical asymptote at $x = -3$.

One way to find the horizontal asymptote is to let $f(x) = y$ and solve for x in terms of y. Then find where the function is undefined for values of y.

$$
\begin{aligned}
y &= \frac{2x - 1}{x + 3} \\
y(x + 3) &= 2x - 1 \\
xy + 3y &= 2x - 1 \\
xy - 2x &= -3y - 1 \\
x(y - 2) &= -3y - 1 \\
x &= \frac{-3y - 1}{y - 2}
\end{aligned}
$$

The rational expression $\frac{-3y - 1}{y - 2}$ is undefined for $y = 2$. Thus, the horizontal asymptote is the line $y = 2$.

Example 2 **Determine the slant asymptote for $f(x) = \frac{3x^2 - 2x + 2}{x - 1}$.**

First use division to rewrite the function.

$$
\begin{array}{r}
3x + 1 \\
x - 1\overline{)3x^2 - 2x + 2} \\
\underline{3x^2 - 3x\phantom{{}+2}} \\
x + 2 \\
\underline{x - 1} \\
3
\end{array}
\quad \to \quad f(x) = 3x + 1 + \frac{3}{x - 1}
$$

As $x \to \infty$, $\frac{3}{x - 1} \to 0$. Therefore, the graph of $f(x)$ will approach that of $y = 3x + 1$. This means that the line $y = 3x + 1$ is a slant asymptote for the graph of $f(x)$.

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Study Guide

Direct, Inverse, and Joint Variation

A **direct variation** can be described by the equation $y = kx^n$. The k in this equation is called the **constant of variation.** To express a direct variation, we say that y varies directly as x^n. An inverse variation can be described by the equation $y = \frac{k}{x^n}$ or $x^n y = k$. When quantities are **inversely proportional,** we say they *vary inversely* with each other. **Joint variation** occurs when one quantity varies directly as the product of two or more other quantities and can be described by the equation $y = kx^n z^n$.

Example 1 **Suppose y varies directly as x and $y = 14$ when $x = 8$.**

a. Find the constant of variation and write an equation of the form $y = kx^n$.

b. Use the equation to find the value of y when $x = 4$.

a. The power of x is 1, so the direct variation equation is $y = kx$.

$y = kx$

$14 = k(8)$ *$y = 14$, $x = 8$*

$1.75 = k$ *Divide each side by 8.*

The constant of variation is 1.75. The equation relating x and y is $y = 1.75x$.

b. $y = 1.75x$

$y = 1.75(4)$ *$x = 4$*

$y = 7$

When $x = 4$, the value of y is 8.

Example 2 **If y varies inversely as x and $y = 102$ when $x = 7$, find x when $y = 12$.**

Use a proportion that relates the values.

$\frac{x_1{}^n}{y_2} = \frac{x_2{}^n}{y_1}$

$\frac{7}{12} = \frac{x}{102}$ *Substitute the known values.*

$12x = 714$ *Cross multiply.*

$x = \frac{714}{12}$ or 59.5   *Divide each side by 12.*

When $y = 12$, the value of x is 59.5.

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Study Guide

Polynomial Functions

The **degree** of a polynomial in one variable is the greatest exponent of its variable. The coefficient of the variable with the greatest exponent is called the **leading coefficient.** If a function $f(x)$ is defined by a polynomial in one variable, then it is a polynomial function. The values of x for which $f(x) = 0$ are called the **zeros** of the function. Zeros of the function are **roots** of the **polynomial equation** when $f(x) = 0$. A polynomial equation of degree n has exactly n roots in the set of complex numbers.

Example 1 **State the degree and leading coefficient of the polynomial function $f(x) = 6x^5 + 8x^3 - 8x$. Then determine whether $\sqrt{\frac{2}{3}}$ is a zero of $f(x)$.**

$6x^5 + 8x^3 - 8x$ has a degree of 5 and a leading coefficient of 6.

Evaluate the function for $x = \sqrt{\frac{2}{3}}$. That is, find $f\left(\sqrt{\frac{2}{3}}\right)$.

$$f\left(\sqrt{\frac{2}{3}}\right) = 6\left(\sqrt{\frac{2}{3}}\right)^5 + 8\left(\sqrt{\frac{2}{3}}\right)^3 - 8\left(\sqrt{\frac{2}{3}}\right) \qquad x = \sqrt{\frac{2}{3}}$$
$$= \frac{24}{9}\sqrt{\frac{2}{3}} + \frac{16}{3}\sqrt{\frac{2}{3}} - 8\sqrt{\frac{2}{3}}$$
$$= 0$$

Since $f\left(\sqrt{\frac{2}{3}}\right) = 0$, $\sqrt{\frac{2}{3}}$ is a zero of $f(x) = 6x^5 + 8x^3 - 8x$.

Example 2 **Write a polynomial equation of least degree with roots 0, $\sqrt{2}i$, and $-\sqrt{2}i$.**

The linear factors for the polynomial are $x - 0$, $x - \sqrt{2}i$, and $x + \sqrt{2}i$. Find the products of these factors.

$$(x - 0)(x - \sqrt{2}i)(x + \sqrt{2}i) = 0$$
$$x(x^2 - 2i^2) = 0$$
$$x(x^2 + 2) = 0 \qquad -2i^2 = -2(-1) \text{ or } 2$$
$$x^3 + 2x = 0$$

Example 3 **State the number of complex roots of the equation $3x^2 + 11x - 4 = 0$. Then find the roots.**

The polynomial has a degree of 2, so there are two complex roots. Factor the equation to find the roots.

$$3x^2 + 11x - 4 = 0$$
$$(3x - 1)(x + 4) = 0$$

To find each root, set each factor equal to zero.

$$3x - 1 = 0 \qquad x + 4 = 0$$
$$3x = 1 \qquad x = -4$$
$$x = \frac{1}{3}$$

The roots are -4 and $\frac{1}{3}$.

4-2

Study Guide

Quadratic Equations

A quadratic equation is a polynomial equation with a degree of 2. Solving quadratic equations by graphing usually does not yield exact answers. Also, some quadratic expressions are not factorable. However, solutions can always be obtained by **completing the square.**

Example 1 **Solve $x^2 - 12x + 7 = 0$ by completing the square.**

$x^2 - 12x + 7 = 0$	
$x^2 - 12x = -7$	*Subtract 7 from each side.*
$x^2 - 12x + 36 = -7 + 36$	*Complete the square by adding* $\left[\frac{1}{2}(-12)\right]^2$, *or 36, to each side.*
$(x - 6)^2 = 29$	*Factor the perfect square trinomial.*
$x - 6 = \pm\sqrt{29}$	*Take the square root of each side.*
$x = 6 \pm \sqrt{29}$	*Add 6 to each side.*

The roots of the equation are $6 \pm \sqrt{29}$.

Completing the square can be used to develop a general formula for solving any quadratic equation of the form $ax^2 + bx + c = 0$. This formula is called the **Quadratic Formula** and can be used to find the roots of any quadratic equation.

Quadratic Formula	If $ax^2 + bx + c = 0$ with $a \neq 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

In the Quadratic Formula, the radicand $b^2 - 4ac$ is called the **discriminant** of the equation. The discriminant tells the nature of the roots of a quadratic equation or the zeros of the related quadratic function.

Example 2 **Find the discriminant of $2x^2 - 3x = 7$ and describe the nature of the roots of the equation. Then solve the equation by using the Quadratic Formula.**

Rewrite the equation using the standard form $ax^2 + bx + c = 0$.

$2x^2 - 3x - 7 = 0$ *$a = 2$, $b = -3$, and $c = -7$*

The value of the discriminant $b^2 - 4ac$ is $(-3)^2 - 4(2)(-7)$, or 65.

Since the value of the discriminant is greater than zero, there are two distinct real roots.

Now substitute the coefficients into the quadratic formula and solve.

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-7)}}{2(2)} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{3 \pm \sqrt{65}}{4}$$

The roots are $\frac{3 + \sqrt{65}}{4}$ and $\frac{3 - \sqrt{65}}{4}$.

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Study Guide

The Remainder and Factor Theorems

The Remainder Theorem	If a polynomial $P(x)$ is divided by $x - r$, the remainder is a constant $P(r)$, and $P(x) = (x - r) \cdot Q(x) + P(r)$ where $Q(x)$ is a polynomial with degree one less than the degree of $P(x)$.

Example 1 **Divide $x^4 - 5x^2 - 17x - 12$ by $x + 3$.**

$$
\begin{array}{r}
x^3 - 3x^2 + 4x - 29 \\
x + 3\,\overline{)\,x^4 + 0x^3 - 5x^2 - 17x - 12} \\
\underline{x^4 + 3x^3} \qquad\qquad\qquad\quad \\
-3x^3 - 5x^2 \qquad\qquad \\
\underline{-3x^3 - 9x^2} \qquad\qquad \\
4x^2 - 17x \qquad \\
\underline{4x^2 + 12x} \qquad \\
-29x - 12 \\
\underline{-29x - 87} \\
75 \leftarrow \textit{remainder}
\end{array}
$$

Find the value of r in this division.

$$
\begin{aligned}
x - r &= x + 3 \\
-r &= 3 \\
r &= -3
\end{aligned}
$$

According to the Remainder Theorem, $P(r)$ or $P(-3)$ should equal 75.

Use the Remainder Theorem to check the remainder found by long division.

$$
\begin{aligned}
P(x) &= x^4 - 5x^2 - 17x - 12 \\
P(-3) &= (-3)^4 - 5(-3)^2 - 17(-3) - 12 \\
&= 81 - 45 + 51 - 12 \text{ or } 75
\end{aligned}
$$

The Factor Theorem is a special case of the Remainder Theorem and can be used to quickly test for factors of a polynomial.

The Factor Theorem	The binomial $x - r$ is a factor of the polynomial $P(x)$ if and only if $P(r) = 0$.

Example 2 **Use the Remainder Theorem to find the remainder when $2x^3 + 5x^2 - 14x - 8$ is divided by $x - 2$. State whether the binomial is a factor of the polynomial. Explain.**

Find $f(2)$ to see if $x - 2$ is a factor.

$$
\begin{aligned}
f(x) &= 2x^3 + 5x^2 - 14x - 8 \\
f(2) &= 2(2)^3 + 5(2)^2 - 14(2) - 8 \\
&= 16 + 20 - 28 - 8 \\
&= 0
\end{aligned}
$$

Since $f(2) = 0$, the remainder is 0. So the binomial $x - 2$ is a factor of the polynomial by the Factor Theorem.

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4-4

Study Guide

The Rational Root Theorem

The **Rational Root Theorem** provides a means of determining possible rational roots of an equation. **Descartes' Rule of Signs** can be used to determine the possible number of positive real zeros and the possible number of negative real zeros.

Rational Root Theorem	Let $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ represent a polynomial equation of degree n with integral coefficients. If a rational number $\frac{p}{q}$, where p and q have no common factors, is a root of the equation, then p is a factor of a_n and q is a factor of a_0.

Example 1 **List the possible rational roots of $x^3 - 5x^2 - 17x - 6 = 0$. Then determine the rational roots.**

p is a factor of 6 and q is a factor of 1

possible values of p: $\pm 1, \pm 2, \pm 3, \pm 6$

possible values of q: ± 1

possible rational roots, $\frac{p}{q}$: $\pm 1, \pm 2, \pm 3, \pm 6$

Test the possible roots using synthetic division.

r	1	−5	−17	−6
1	1	−4	−21	−27
−1	1	−6	−11	5
2	1	−3	−23	−52
−2	1	−7	−3	0
3	1	−2	−23	−75
−3	1	−8	7	−27
6	1	1	−11	−72
−6	1	−11	49	−300

← *There is a root at $x = -2$.*
The depressed polynomial is $x^2 - 7x - 3$. You can use the Quadratic Formula to find the two irrational roots.

Example 2 **Find the number of possible positive real zeros and the number of possible negative real zeros for $f(x) = 4x^4 - 13x^3 - 21x^2 + 38x - 8$.**

According to Descartes' Rule of Signs, the number of positive real zeros is the same as the number of sign changes of the coefficients of the terms in descending order or is less than this by an even number. Count the sign changes.

$$f(x) = 4x^4 - 13x^3 - 21x^2 + 38x - 8$$
$$4 \quad -13 \quad -21 \quad 38 \quad -8$$

There are three changes. So, there are 3 or 1 positive real zeros.

The number of negative real zeros is the same as the number of sign changes of the coefficients of the terms of $f(-x)$, or less than this number by an even number.

$$f(-x) = 4(-x)^4 - 13(-x)^3 - 21(-x)^2 + 38(-x) - 8$$
$$4 \quad 13 \quad -21 \quad -38 \quad -8$$

There is one change. So, there is 1 negative real zero.

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Study Guide

Locating Zeros of a Polynomial Function

A polynomial function may have real zeros that are not rational numbers. The **Location Principle** provides a means of locating and approximating real zeros. For the polynomial function $y = f(x)$, if a and b are two numbers with $f(a)$ positive and $f(b)$ negative, then there must be at least one real zero between a and b. For example, if $f(x) = x^2 - 2$, $f(0) = -2$ and $f(2) = 2$. Thus, a zero exists somewhere between 0 and 2.

The **Upper Bound Theorem** and the **Lower Bound Theorem** are also useful in locating the zeros of a function and in determining whether all the zeros have been found. If a polynomial function $P(x)$ is divided by $x - c$, and the quotient and the remainder have no change in sign, c is an **upper bound** of the zeros of $P(x)$. If c is an upper bound of the zeros of $P(-x)$, then $-c$ is a **lower bound** of the zeros of $P(x)$.

Example 1 **Determine between which consecutive integers the real zeros of $f(x) = x^3 - 2x^2 - 4x + 5$ are located.**

According to Descartes' Rule of Signs, there are two or zero positive real roots and one negative real root. Use synthetic division to evaluate $f(x)$ for consecutive integral values of x.

r	1	−2	−4	5
−4	1	−6	20	−75
−3	1	−5	11	−28
−2	1	−4	4	−3
−1	1	−3	−1	6
0	1	−2	−4	5
1	1	−1	−5	0
2	1	0	−4	−3
3	1	1	−1	2

There is a zero at 1. The changes in sign indicate that there are also zeros between −2 and −1 and between 2 and 3. This result is consistent with Descartes' Rule of Signs.

Example 2 **Use the Upper Bound Theorem to show that 3 is an upper bound and the Lower Bound Theorem to show that −2 is a lower bound of the zeros of $f(x) = x^3 - 3x^2 + x - 1$.**

Synthetic division is the most efficient way to test potential upper and lower bounds. First, test for the upper bound.

r	1	−3	1	−1
3	1	0	1	2

Since there is no change in the signs in the quotient and remainder, 3 is an upper bound.
Now, test for the lower bound of $f(x)$ by showing that 2 is an upper bound of $f(-x)$.

$$f(-x) = (-x)^3 - 3(-x)^2 + (-x) - 1 = -x^3 - 3x^2 - x - 1$$

r	−1	−3	−1	−1
2	−1	−5	−11	−23

Since there is no change in the signs, −2 is a lower bound of $f(x)$.

Study Guide

Rational Equations and Partial Fractions

A **rational equation** consists of one or more rational expressions. One way to solve a rational equation is to multiply each side of the equation by the least common denominator (LCD). Any possible solution that results in a zero in the denominator must be excluded from your list of solutions. In order to find the LCD, it is sometimes necessary to factor the denominators. If a denominator can be factored, the expression can be rewritten as the sum of **partial fractions.**

Example 1 **Solve $\frac{x+1}{3(x-2)} = \frac{5x}{6} + \frac{1}{x-2}$.**

$$6(x-2)\left[\frac{x+1}{3(x-2)}\right] = 6(x-2)\left(\frac{5x}{6} + \frac{1}{x-2}\right)$$ *Multiply each side by the LCD, 6(x − 2).*

$$2(x+1) = (x-2)(5x) + 6(1)$$

$$2x + 2 = 5x^2 - 10x + 6$$ *Simplify.*

$$5x^2 - 12x + 4 = 0$$ *Write in standard form.*

$$(5x-2)(x-2) = 0$$ *Factor.*

$$5x - 2 = 0 \qquad x - 2 = 0$$

$$x = \frac{2}{5} \qquad x = 2$$

Since x cannot equal 2 because a zero denominator results, the only solution is $\frac{2}{5}$.

Example 2 **Decompose $\frac{2x-1}{x^2+2x-3}$ into partial fractions.**

Factor the denominator and express the factored form as the sum of two fractions using A and B as numerators and the factors as denominators.

$$x^2 + 2x - 3 = (x-1)(x+3)$$

$$\frac{2x-1}{x^2+2x-3} = \frac{A}{x-1} + \frac{B}{x+3}$$

$$2x - 1 = A(x+3) + B(x-1)$$

Let $x = 1$.
$$2(1) - 1 = A(1+3)$$
$$1 = 4A$$
$$A = \frac{1}{4}$$

Let $x = -3$.
$$2(-3) - 1 = B(-3-1)$$
$$-7 = -4B$$
$$B = \frac{7}{4}$$

$$\frac{2x-1}{x^2+2x-3} = \frac{\frac{1}{4}}{x-1} + \frac{\frac{7}{4}}{x+3} \text{ or } \frac{1}{4(x-1)} + \frac{7}{4(x+3)}$$

Example 3 **Solve $\frac{1}{2t} + \frac{3}{4t} > 1$.**

Rewrite the inequality as the related function $f(t) = \frac{1}{2t} + \frac{3}{4t} - 1$.

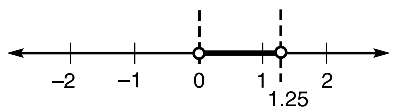
Find the zeros of this function.

$$4t\left(\frac{1}{2t}\right) + 4t\left(\frac{3}{4t}\right) - 4t(1) = 4t(0)$$

$$5 - 4t = 0$$

$$t = 1.25$$

The zero is 1.25. The excluded value is 0. On a number line, mark these values with vertical dashed lines. Testing each interval shows the solution set to be $0 < t < 1.25$.



4-7

Study Guide

Radical Equations and Inequalities

Equations in which radical expressions include variables are known as **radical equations.** To solve radical equations, first isolate the radical on one side of the equation. Then raise each side of the equation to the proper power to eliminate the radical expression. This process of raising each side of an equation to a power often introduces **extraneous solutions.** Therefore, it is important to check all possible solutions in the original equation to determine if any of them should be eliminated from the solution set. **Radical inequalities** are solved using the same techniques used for solving radical equations.

Example 1 **Solve $3 = \sqrt[3]{x^2 - 2x + 1} - 1$.**

$3 = \sqrt[3]{x^2 - 2x + 1} - 1$

$4 = \sqrt[3]{x^2 - 2x + 1}$ *Isolate the cube root.*

$64 = x^2 - 2x + 1$ *Cube each side.*

$0 = x^2 - 2x - 63$

$0 = (x - 9)(x + 7)$ *Factor.*

$x - 9 = 0$ $\quad$ $x + 7 = 0$

$x = 9$ $\quad$ $x = -7$

Check both solutions to make sure they are not extraneous.

$x = 9$:

$3 = \sqrt[3]{x^2 - 2x + 1} - 1$

$3 \stackrel{?}{=} \sqrt[3]{(9)^2 - 2(9) + 1} - 1$

$3 \stackrel{?}{=} \sqrt[3]{64} - 1$

$3 \stackrel{?}{=} 4 - 1$

$3 = 3$ ✔

$x = -7$:

$3 = \sqrt[3]{x^2 - 2x + 1} - 1$

$3 \stackrel{?}{=} \sqrt[3]{(-7)^2 - 2(-7) + 1} - 1$

$3 \stackrel{?}{=} \sqrt[3]{64} - 1$

$3 \stackrel{?}{=} 4 - 1$

$3 = 3$ ✔

Example 2 **Solve $2\sqrt{3x + 5} > 2$.**

$2\sqrt{3x + 5} > 2$

$4(3x + 5) > 4$ *Square each side.*

$3x + 5 > 1$ *Divide each side by 4.*

$3x > -4$

$x > -1.33$

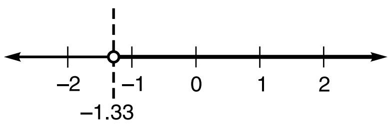
In order for $\sqrt{3x + 5}$ to be a real number, $3x + 5$ must be greater than or equal to zero.

$3x + 5 \geq 0$

$3x \geq -5$

$x \geq -1.67$

Since -1.33 is greater than -1.67, the solution is $x > -1.33$. Check this solution by testing values in the intervals defined by the solution. Then graph the solution on a number line.



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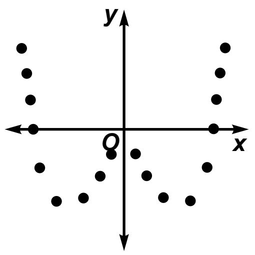
4-8 Study Guide

Modeling Real-World Data with Polynomial Functions

In order to model real-world data using polynomial functions, you must be able to identify the general shape of the graph of each type of polynomial function.

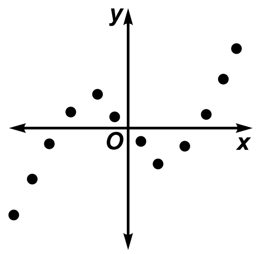
Example 1 **Determine the type of polynomial function that could be used to represent the data in each scatter plot.**

a.



The scatter plot seems to change direction three times, so a quartic function would best fit the scatter plot.

b.

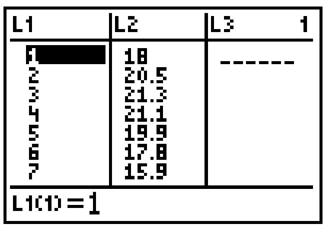


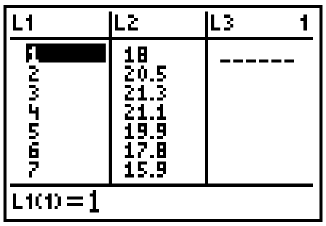
The scatter plot seems to change direction two times, so a cubic function would best fit the scatter plot.

Example 2 **An oil tanker collides with another ship and starts leaking oil. The Coast Guard measures the rate of flow of oil from the tanker and obtains the data shown in the table. Use a graphing calculator to write a polynomial function to model the set of data.**

Time (hours)	Flow rate (100s of liters per hour)
1	18.0
2	20.5
3	21.3
4	21.1
5	19.9
6	17.8
7	15.9
8	11.3
9	7.6
10	3.7

Clear the statistical memory and input the data. Adjust the window to an appropriate setting and graph the statistical data. The data appear to change direction one time, so a quadratic function will fit the scatter plot. Press STAT, highlight **CALC**, and choose **5:QuadReg**. Then enter 2nd **[L1]** , 2nd **[L2]** ENTER. Rounding the coefficients to the nearest tenth, $f(x) = -0.4x^2 + 2.8x + 16.3$ models the data. Since the value of the coefficient of determination r^2 is very close to 1, the polynomial is an excellent fit.





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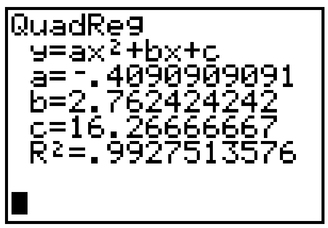


5-1

Study Guide

Angles and Degree Measure

Decimal degree measures can be expressed in **degrees(°)**, **minutes(′)**, and **seconds(″)**.

Example 1 **a. Change 12.520° to degrees, minutes, and seconds.**

$12.520° = 12° + (0.520 \cdot 60)'$ *Multiply the decimal portion of*
$= 12° + 31.2'$ *the degrees by 60 to find minutes.*
$= 12° + 31' + (0.2 \cdot 60)''$ *Multiply the decimal portion of*
$= 12° + 31' + 12''$ *the minutes by 60 to find seconds.*

12.520° can be written as 12° 31′ 12″.

b. Write 24° 15′ 33″ as a decimal rounded to the nearest thousandth.

$$24°\ 15'\ 33'' = 24° + 15'\left(\frac{1°}{60'}\right) + 33''\left(\frac{1°}{3600''}\right)$$
$$= 24.259°$$

24° 15′ 33″ can be written as 24.259°.

An angle may be generated by the rotation of one ray multiple times about the origin.

Example 2 **Give the angle measure represented by each rotation.**

a. 2.3 rotations clockwise

$2.3 \times -360 = -828$ *Clockwise rotations have negative measures.*

The angle measure of 2.3 clockwise rotations is −828°.

b. 4.2 rotations counterclockwise

$4.2 \times 360 = 1512$ *Counterclockwise rotations have positive measures.*

The angle measure of 4.2 counterclockwise rotations is 1512°.

If α is a nonquadrantal angle in standard position, its **reference angle** is defined as the acute angle formed by the terminal side of the given angle and the x-axis.

Reference Angle Rule	For any angle α, $0° < \alpha < 360°$, its reference angle α' is defined by a. α, when the terminal side is in Quadrant I, b. $180° - \alpha$, when the terminal side is in Quadrant II, c. $\alpha - 180°$, when the terminal side is in Quadrant III, and d. $360° - \alpha$, when the terminal side is in Quadrant IV.

Example 3 **Find the measure of the reference angle for 220°.**

Because 220° is between 180° and 270°, the terminal side of the angle is in Quadrant III.

$220° - 180° = 40°$

The reference angle is 40°.

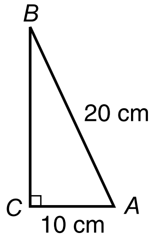
5-2

Study Guide

Trigonometric Ratios in Right Triangles

The ratios of the sides of right triangles can be used to define the **trigonometric** ratios known as the **sine, cosine,** and **tangent.**

Example 1 **Find the values of the sine, cosine, and tangent for $\angle A$.**



First find the length of $\overline{BC}$.

$(AC)^2 + (BC)^2 = (AB)^2$ *Pythagorean Theorem*

$10^2 + (BC)^2 = 20^2$ *Substitute 10 for AC and 20 for AB.*

$(BC)^2 = 300$

$BC = \sqrt{300}$ or $10\sqrt{3}$ *Take the square root of each side. Disregard the negative root.*

Then write each trigonometric ratio.

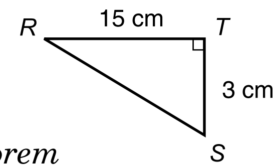
$\sin A = \frac{\textit{side opposite}}{\textit{hypotenuse}}$ $\cos A = \frac{\textit{side adjacent}}{\textit{hypotenuse}}$ $\tan A = \frac{\textit{side opposite}}{\textit{side adjacent}}$

$\sin A = \frac{10\sqrt{3}}{20}$ or $\frac{\sqrt{3}}{2}$ $\cos A = \frac{10}{20}$ or $\frac{1}{2}$ $\tan A = \frac{10\sqrt{3}}{10}$ or $\sqrt{3}$

Trigonometric ratios are often simplified but never written as mixed numbers.

Three other trigonometric ratios, called **cosecant, secant,** and **cotangent,** are reciprocals of sine, cosine, and tangent, respectively.

Example 2 **Find the values of the six trigonometric ratios for $\angle R$.**



First determine the length of the hypotenuse.

$(RT)^2 + (ST)^2 = (RS)^2$ *Pythagorean Theorem*

$15^2 + 3^2 = (RS)^2$ *RT = 15, ST = 3*

$(RS)^2 = 234$

$RS = \sqrt{234}$ or $3\sqrt{26}$ *Disregard the negative root.*

$\sin R = \frac{\textit{side opposite}}{\textit{hypotenuse}}$ $\cos R = \frac{\textit{side adjacent}}{\textit{hypotenuse}}$ $\tan R = \frac{\textit{side opposite}}{\textit{side adjacent}}$

$\sin R = \frac{3}{3\sqrt{26}}$ or $\frac{\sqrt{26}}{26}$ $\cos R = \frac{15}{3\sqrt{26}}$ or $\frac{5\sqrt{26}}{26}$ $\tan R = \frac{3}{15}$ or $\frac{1}{5}$

$\csc R = \frac{\textit{hypotenuse}}{\textit{side opposite}}$ $\sec R = \frac{\textit{hypotenuse}}{\textit{side adjacent}}$ $\cot R = \frac{\textit{side adjacent}}{\textit{side opposite}}$

$\csc R = \frac{3\sqrt{26}}{3}$ or $\sqrt{26}$ $\sec R = \frac{3\sqrt{26}}{15}$ or $\frac{\sqrt{26}}{5}$ $\cot R = \frac{15}{3}$ or 5

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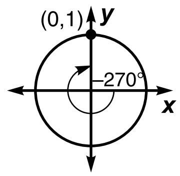
Study Guide

Trigonometric Functions on the Unit Circle

Example 1 **Use the unit circle to find cot $(-270°)$.**



The terminal side of a $-270°$ angle in standard position is the positive y-axis, which intersects the unit circle at $(0, 1)$.

By definition, $\cot(-270°) = \frac{x}{y}$ or $\frac{0}{1}$.

Therefore, $\cot(-270°) = 0$.

Trigonometric Functions of an Angle in Standard Position	$\sin\theta = \frac{y}{r}$	$\cos\theta = \frac{x}{r}$	$\tan\theta = \frac{y}{x}$
	$\csc\theta = \frac{r}{y}$	$\sec\theta = \frac{r}{x}$	$\cot\theta = \frac{x}{y}$

Example 2 **Find the values of the six trigonometric functions for angle θ in standard position if a point with coordinates $(-9, 12)$ lies on its terminal side.**

We know that $x = -9$ and $y = 12$. We need to find r.

$r = \sqrt{x^2 + y^2}$ *Pythagorean Theorem*

$r = \sqrt{(-9)^2 + 12^2}$ *Substitute −9 for x and 12 for y.*

$r = \sqrt{225}$ or 15 *Disregard the negative root.*

$\sin\theta = \frac{12}{15}$ or $\frac{4}{5}$ $\cos\theta = \frac{-9}{15}$ or $-\frac{3}{5}$ $\tan\theta = \frac{12}{-9}$ or $-\frac{4}{3}$

$\csc\theta = \frac{15}{12}$ or $\frac{5}{4}$ $\sec\theta = \frac{15}{-9}$ or $-\frac{5}{3}$ $\cot\theta = \frac{-9}{12}$ or $-\frac{3}{4}$

Example 3 **Suppose θ is an angle in standard position whose terminal side lies in Quadrant I. If $\cos\theta = \frac{3}{5}$, find the values of the remaining five trigonometric functions of θ.**

$r^2 = x^2 + y^2$ *Pythagorean Theorem*

$5^2 = 3^2 + y^2$ *Substitute 5 for r and 3 for x.*

$16 = y^2$

$\pm 4 = y$ *Take the square root of each side.*

Since the terminal side of θ lies in Quadrant I, y must be positive.

$\sin\theta = \frac{4}{5}$ $\tan\theta = \frac{4}{3}$

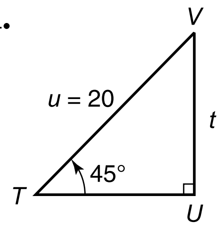
$\csc\theta = \frac{5}{4}$ $\sec\theta = \frac{5}{3}$ $\cot\theta = \frac{3}{4}$

Study Guide

Applying Trigonometric Functions

Trigonometric functions can be used to solve problems involving right triangles.

Example 1 **If $T = 45°$ and $u = 20$, find t to the nearest tenth.**



From the figure, we know the measures of an angle and the hypotenuse. We want to know the measure of the side opposite the given angle. The sine function relates the side opposite the angle and the hypotenuse.

$\sin T = \frac{t}{u}$ *$\sin = \frac{\text{side opposite}}{\text{hypotenuse}}$*

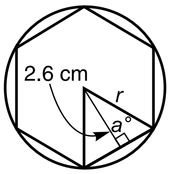
$\sin 45° = \frac{t}{20}$ *Substitute 45° for T and 20 for u.*

$20 \sin 45° = t$ *Multiply each side by 20.*

$14.14213562 \approx t$ *Use a calculator.*

Therefore, t is about 14.1.

Example 2 ***Geometry*** **The apothem of a regular polygon is the measure of a line segment from the center of the polygon to the midpoint of one of its sides. The apothem of a regular hexagon is 2.6 centimeters. Find the radius of the circle circumscribed about the hexagon to the nearest tenth.**



First draw a diagram. Let a be the angle measure formed by a radius and its adjacent apothem. The measure of a is $360° \div 12$ or $30°$. Now we know the measures of an angle and the side adjacent to the angle.

$\cos 30° = \frac{2.6}{r}$ *$\cos = \frac{\text{side adjacent}}{\text{hypotenuse}}$*

$r \cos 30° = 2.6$ *Multiply each side by r.*

$r = \frac{2.6}{\cos 30°}$ *Divide each side by cos 30°.*

$r \approx 3.0022214$ *Use a calculator.*

Therefore, the radius is about 3.0 centimeters.

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Study Guide

Solving Right Triangles

When we know a trigonometric value of an angle but not the value of the angle, we need to use the inverse of the trigonometric function.

Trigonometric Function	Inverse Trigonometric Relation
$y = \sin x$	$x = \sin^{-1} y$ or $x = \arcsin y$
$y = \cos x$	$x = \cos^{-1} y$ or $x = \arccos y$
$y = \tan x$	$x = \tan^{-1} y$ or $x = \arctan y$

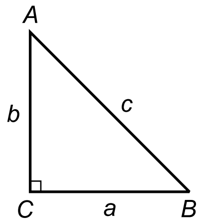
Example 1 **Solve $\tan x = \sqrt{3}$.**

If $\tan x = \sqrt{3}$, then x is an angle whose tangent is $\sqrt{3}$.

$x = \arctan \sqrt{3}$

From a table of values, you can determine that x equals 60°, 240°, or any angle coterminal with these angles.

Example 2 **If $c = 22$ and $b = 12$, find B.**



In this problem, we know the side opposite the angle and the hypotenuse. The sine function relates the side opposite the angle and the hypotenuse.

$\sin B = \frac{b}{c}$ — *$\sin = \frac{\text{side opposite}}{\text{hypotenuse}}$*

$\sin B = \frac{12}{22}$ — *Substitute 12 for b and 22 for c.*

$B = \sin^{-1}\left(\frac{12}{22}\right)$ — *Definition of inverse*

$B \approx 33.05573115$ or about 33.1°.

Example 3 **Solve the triangle where $b = 20$ and $c = 35$, given the triangle above.**

$$a^2 + b^2 = c^2$$
$$a^2 + 20^2 = 35^2$$
$$a = \sqrt{825}$$
$$a \approx 28.72281323$$

$$\cos A = \frac{b}{c}$$
$$\cos A = \frac{20}{35}$$
$$A = \cos^{-1}\left(\frac{20}{35}\right)$$
$$A \approx 55.15009542$$

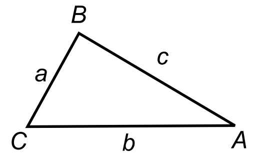
$$55.15009542 + B \approx 90$$
$$B \approx 34.84990458$$

Therefore, $a \approx 28.7$, $A \approx 55.2°$, and $B \approx 34.8°$.

Study Guide

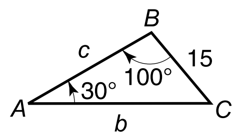
The Law of Sines

Given the measures of two angles and one side of a triangle, we can use the **Law of Sines** to find one unique solution for the triangle.



Law of Sines	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Example 1 **Solve $\triangle ABC$ if $A = 30°$, $B = 100°$, and $a = 15$.**



First find the measure of $\angle C$.
$C = 180° - (30° + 100°)$ or $50°$

Use the Law of Sines to find b and c.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{15}{\sin 30°} = \frac{b}{\sin 100°} \qquad \frac{c}{\sin 50°} = \frac{15}{\sin 30°}$$

$$\frac{15 \sin 100°}{\sin 30°} = b \qquad c = \frac{15 \sin 50°}{\sin 30°}$$

$$29.54423259 \approx b \qquad c \approx 22.98133329$$

Therefore, $C = 50°$, $b \approx 29.5$, and $c \approx 23.0$.

The area of any triangle can be expressed in terms of two sides of a triangle and the measure of the included angle.

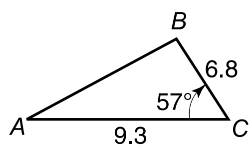
Area (K) of a Triangle	$K = \frac{1}{2}bc \sin A$ $K = \frac{1}{2}ac \sin B$ $K = \frac{1}{2}ab \sin C$

Example 2 **Find the area of $\triangle ABC$ if $a = 6.8$, $b = 9.3$, and $C = 57°$.**

$K = \frac{1}{2}ab \sin C$

$K = \frac{1}{2}(6.8)(9.3) \sin 57°$

$K \approx 26.51876336$



The area of $\triangle ABC$ is about 26.5 square units.

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Study Guide

The Ambiguous Case for the Law of Sines

Case 1: $A < 90°$ for a, b, and A	
$a < b \sin A$	no solution
$a = b \sin A$	one solution
$a \geq b$	one solution
$b \sin A < a < b$	two solutions
Case 2: $A \geq 90°$	
$a \leq b$	no solution
$a > b$	one solution

If we know the measures of two sides and a nonincluded angle of a triangle, three situations are possible: no triangle exists, exactly one triangle exists, or two triangles exist. A triangle with two solutions is called the **ambiguous case**.

Example **Find all solutions for the triangle if $a = 20$, $b = 30$, and $A = 40°$. If no solutions exist, write *none*.**

Since $40° < 90°$, consider Case 1.
$b \sin A = 30 \sin 40°$
$b \sin A \approx 19.28362829$
Since $19.3 < 20 < 30$, there are two solutions for the triangle.
Use the Law of Sines to find B.

$$\frac{20}{\sin 40°} = \frac{30}{\sin B} \qquad \frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\sin B = \frac{30 \sin 40°}{20}$$

$$B = \sin^{-1}\left(\frac{30 \sin 40°}{20}\right)$$

$$B \approx 74.61856831$$

So, $B \approx 74.6°$. Since we know there are two solutions, there must be another possible measurement for B. In the second case, B must be less than 180° and have the same sine value. Since we know that if $\alpha < 90$, $\sin \alpha = \sin(180 - \alpha)$, $180° - 74.6°$ or $105.4°$ is another possible measure for B. Now solve the triangle for each possible measure of B.

Solution I

$C \approx 180° - (40° + 74.6°)$ or $65.4°$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{20}{\sin 40°} \approx \frac{c}{\sin 65.4°}$$

$$c \approx \frac{20 \sin 65.4°}{\sin 40°}$$

$$c \approx 28.29040558$$

One solution is $B \approx 74.6°$, $C \approx 65.4°$, and $c \approx 28.3$.

Solution II

$C \approx 180° - (40° + 105.4°)$ or $34.6°$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{20}{\sin 40°} \approx \frac{c}{\sin 34.6°}$$

$$c \approx \frac{20 \sin 34.6°}{\sin 40°}$$

$$c \approx 17.66816088$$

Another solution is $B \approx 105.4°$, $C \approx 34.6°$, and $c \approx 17.7$.

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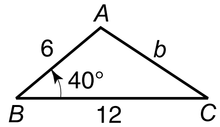
Study Guide

The Law of Cosines

When we know the measures of two sides of a triangle and the included angle, we can use the **Law of Cosines** to find the measure of the third side. Often times we will use both the Law of Cosines and the Law of Sines to solve a triangle.

Law of Cosines	$a^2 = b^2 + c^2 - 2bc \cos A$ $b^2 = a^2 + c^2 - 2ac \cos B$ $c^2 = a^2 + b^2 - 2ab \cos C$

Example 1 **Solve $\triangle ABC$ if $B = 40°$, $a = 12$, and $c = 6$.**



$b^2 = a^2 + c^2 - 2ac \cos B$ *Law of Cosines*
$b^2 = 12^2 + 6^2 - 2(12)(6) \cos 40°$
$b^2 \approx 69.68960019$
$b \approx 8.348029719$
So, $b \approx 8.34$

$\frac{b}{\sin B} = \frac{c}{\sin C}$ *Law of Sines*

$\frac{8.3}{\sin 40°} \approx \frac{6}{\sin C}$

$\sin C \approx \frac{6 \sin 40°}{8.3}$

$C \approx \sin^{-1}\left(\frac{6 \sin 40°}{8.3}\right)$

$C \approx 27.68859159$

So, $C \approx 27.7°$.
$A \approx 180° - (40° + 27.7°) \approx 112.3°$

The solution of this triangle is $b \approx 8.3$, $A \approx 112.3°$, and $C \approx 27.7°$.

Example 2 **Find the area of $\triangle ABC$ if $a = 5$, $b = 8$, and $c = 10$.**

First, find the semiperimeter of $\triangle ABC$.
$s = \frac{1}{2}(a + b + c)$

$s = \frac{1}{2}(5 + 8 + 10)$

$s = 11.5$

Now, apply Hero's Formula
$k = \sqrt{s(s-a)(s-b)(s-c)}$
$k = \sqrt{11.5(11.5-5)(11.5-8)(11.5-10)}$
$k = \sqrt{392.4375}$
$k = \approx 19.81003534$

The area of the triangle is about 19.8 square units.

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Study Guide

Angles and Radian Measure

An angle of one complete revolution can be represented either by 360° or by 2π radians. Thus, the following formulas can be used to relate degree and **radian** measures.

Degree/Radian Conversion Formulas	1 radian $= \frac{180}{\pi}$ degrees or about 57.3° 1 degree $= \frac{\pi}{180}$ radians or about 0.017 radian

Example 1 **a. Change 36° to radian measure in terms of π.**
b. Change $-\frac{17\pi}{3}$ radians to degree measure.

a. $36° = 36° \times \frac{\pi}{180°}$
$= \frac{\pi}{5}$

b. $-\frac{17\pi}{3} = -\frac{17\pi}{3} \times \frac{180°}{\pi}$
$= -1020°$

Example 2 **Evaluate $\sin \frac{3\pi}{4}$.**

The reference angle for $\frac{3\pi}{4}$ is $\frac{\pi}{4}$. Since $\frac{\pi}{4} = 45°$, the terminal side of the angle intersects the unit circle at a point with coordinates of $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Because the terminal side of $\frac{3\pi}{4}$ lies in Quadrant II, the x-coordinate is negative and the y-coordinate is positive. Therefore, $\sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}$.

Example 3 **Given a central angle of 147°, find the length of its intercepted arc in a circle of radius 10 centimeters. Round to the nearest tenth.**

First convert the measure of the central angle from degrees to radians.

$147° = 147° \times \frac{\pi}{180°}$ *1 degree* $= \frac{\pi}{180°}$
$= \frac{49\pi}{60}$

Then find the length of the arc.

$s = r\theta$ *Formula for the length of an arc*

$s = 10\left(\frac{49\pi}{60}\right)$ $r = 10$, $\theta = \frac{49\pi}{60}$

$s \approx 25.65634$

The length of the arc is about 25.7 cm.

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Study Guide

Linear and Angular Velocity

As a circular object rotates about its center, an object at the edge moves through an angle relative to the object's starting position. That is known as the **angular displacement,** or angle of rotation. **Angular velocity** ω is given by $\omega = \frac{\theta}{t}$, where θ is the angular displacement in radians and t is time. **Linear velocity** v is given by $v = r\frac{\theta}{t}$, where $\frac{\theta}{t}$ represents the angular velocity in radians per unit of time. Since $\omega = \frac{\theta}{t}$, this formula can also be written as $v = r\omega$.

Example 1 **Determine the angular displacement in radians of 3.5 revolutions. Round to the nearest tenth.**

Each revolution equals 2π radians. For 3.5 revolutions, the number of radians is $3.5 \times 2\pi$, or 7π. 7π radians equals about 22.0 radians.

Example 2 **Determine the angular velocity if 8.2 revolutions are completed in 3 seconds. Round to the nearest tenth.**

The angular displacement is $8.2 \times 2\pi$, or 16.4π radians.

$\omega = \frac{\theta}{t}$

$\omega = \frac{16.4\pi}{3}$ *$\theta = 16.4\pi$, $t = 3$*

$\omega \approx 17.17403984$ *Use a calculator.*

The angular velocity is about 17.2 radians per second.

Example 3 **Determine the linear velocity of a point rotating at an angular velocity of 13π radians per second at a distance of 7 centimeters from the center of the rotating object. Round to the nearest tenth.**

$v = r\omega$

$v = 7(13\pi)$ *$r = 7$, $\omega = 13\pi$*

$v \approx 285.8849315$ *Use a calculator.*

The linear velocity is about 285.9 centimeters per second.

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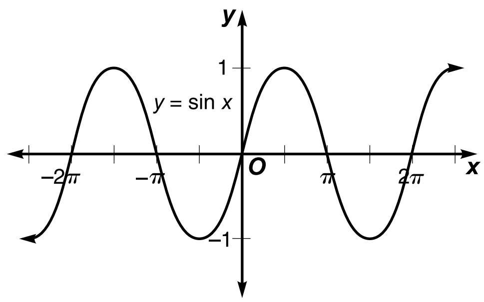
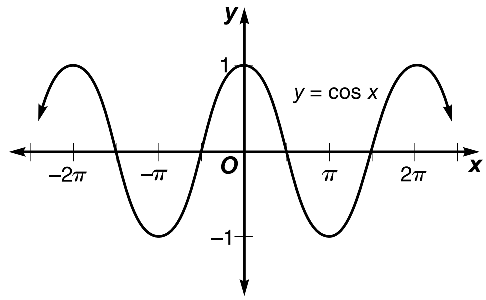
Study Guide

Graphing Sine and Cosine Functions

If the values of a function are the same for each given interval of the domain, the function is said to be **periodic.** Consider the graphs of $y = \sin x$ and $y = \cos x$ shown below. Notice that for both graphs the period is 2π and the range is from -1 to 1, inclusive.





Properties of the Graph of $y = \sin x$	Properties of the Graph of $y = \cos x$
The x-intercepts are located at πn, where n is an integer.	The x-intercepts are located at $\frac{\pi}{2} + \pi n$, where n is an integer.
The y-intercept is 0.	The y-intercept is 1.
The maximum values are $y = 1$ and occur when $x = \frac{\pi}{2} + 2\pi n$, when n is an integer.	The maximum values are $y = 1$ and occur when $x = \pi n$, where n is an even integer.
The minimum values are $y = -1$ and occur when $x = \frac{3\pi}{2} + 2\pi n$, where n is an integer.	The minimum values are $y = -1$ and occur when $x = \pi n$, where n is an odd integer.

Example 1 **Find $\sin \frac{7\pi}{2}$ by referring to the graph of the sine function.**

The period of the sine function is 2π. Since $\frac{7\pi}{2} > 2\pi$, rewrite $\frac{7\pi}{2}$ as a sum involving 2π.

$\frac{7\pi}{2} = 2\pi(1) + \frac{3\pi}{2}$ *This is a form of* $\frac{3\pi}{2} + 2\pi n$.

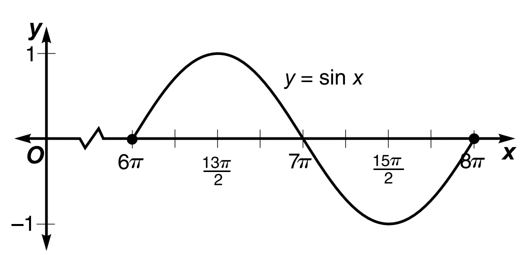
So, $\sin \frac{7\pi}{2} = \sin \frac{3\pi}{2}$ or -1.

Example 2 **Find the values of θ for which $\cos \theta = 0$ is true.**

Since $\cos \theta = 0$ indicates the x-intercepts of the cosine function, $\cos \theta = 0$ if $\theta = \frac{\pi}{2} + \pi n$, where n is an integer.

Example 3 **Graph $y = \sin x$ for $6\pi \leq x \leq 8\pi$.**

The graph crosses the x-axis at 6π, 7π, and 8π. Its maximum value of 1 is at $x = \frac{13\pi}{2}$, and its minimum value of -1 is at $x = \frac{15\pi}{2}$. Use this information to sketch the graph.



Study Guide

Amplitude and Period of Sine and Cosine Functions

The **amplitude** of the functions $y = A \sin \theta$ and $y = A \cos \theta$ is the absolute value of A, or $|A|$. The period of the functions $y = \sin k\theta$ and $y = \cos k\theta$ is $\frac{2\pi}{k}$, where $k > 0$.

Example 1 **State the amplitude and period for the function $y = -2 \cos \frac{\theta}{4}$.**

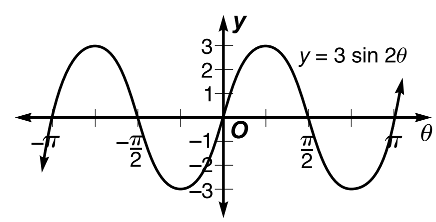
The definition of *amplitude* states that the amplitude of $y = A \cos \theta$ is $|A|$. Therefore, the amplitude of $y = -2 \cos \frac{\theta}{4}$ is $|-2|$, or 2.

The definition of *period* states that the period of $y = \cos k\theta$ is $\frac{2\pi}{k}$. Since $-2 \cos \frac{\theta}{4}$ equals $-2 \cos \left(\frac{1}{4}\theta\right)$, the period is $\frac{2\pi}{\frac{1}{4}}$ or 8π.

Example 2 **State the amplitude and period for the function $y = 3 \sin 2\theta$. Then graph the function.**

Since $A = 3$, the amplitude is $|3|$ or 3.
Since $k = 2$, the period is $\frac{2\pi}{2}$ or π.

Use the amplitude and period above and the basic shape of the sine function to graph the equation.



Example 3 **Write an equation of the sine function with amplitude 6.7 and period 3π.**

The form of the equation will be $y = A \sin k\theta$. First find the possible values of A for an amplitude of 6.7.

$|A| = 6.7$
$A = 6.7$ or -6.7

Since there are two values of A, two possible equations exist.

Now find the value of k when the period is 3π.

$\frac{2\pi}{k} = 3\pi$ *The period of the sine function is $\frac{2\pi}{k}$.*
$k = \frac{2\pi}{3\pi}$ or $\frac{2}{3}$

The possible equations are $y = 6.7 \sin \frac{2}{3}\theta$ or $y = -6.7 \sin \frac{2}{3}\theta$.

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6-5 Study Guide

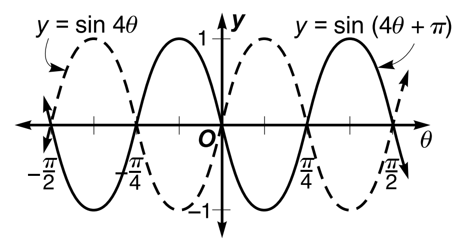
Translations of Sine and Cosine Functions

A horizontal translation of a trigonometric function is called a **phase shift**. The phase shift of the functions $y = A \sin (k\theta + c)$ and $y = A \cos (k\theta + c)$ is $-\frac{c}{k}$, where $k > 0$. If $c > 0$, the shift is to the left. If $c < 0$, the shift is to the right. The **vertical shift** of the functions $y = A \sin (k\theta + c) + h$ and $y = A \cos (k\theta + c) + h$ is h. If $h > 0$, the shift is upward. If $h < 0$, the shift is downward. The **midline** about which the graph oscillates is $y = h$.

Example 1 **State the phase shift for $y = \sin (4\theta + \pi)$. Then graph the function.**

The phase shift of the function is $-\frac{c}{k}$ or $-\frac{\pi}{4}$.

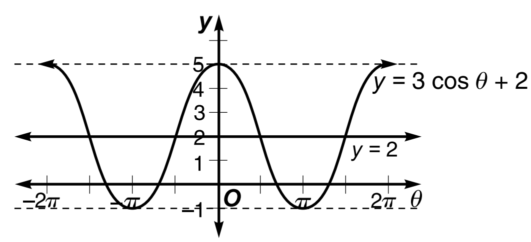
To graph $y = \sin (4\theta + \pi)$, consider the graph of $y = \sin 4\theta$. The graph of $y = \sin 4\theta$ has an amplitude of 1 and a period of $\frac{\pi}{2}$. Graph this function, then shift the graph $-\frac{\pi}{4}$.



Example 2 **State the vertical shift and the equation of the midline for $y = 3 \cos \theta + 2$. Then graph the function.**

The vertical shift is 2 units upward. The midline is the graph of $y = 2$.

To graph the function, draw the midline. Since the amplitude of the function is $|3|$, or 3, draw dashed lines parallel to the midline which are 3 units above and below $y = 2$. That is, $y = 5$ and $y = -1$. Then draw the cosine curve with a period of 2π.



Example 3 **Write an equation of the cosine function with amplitude 2.9, period $\frac{2\pi}{5}$, phase shift $-\frac{\pi}{2}$, and vertical shift -3.**

The form of the equation will be $y = A \cos (k\theta + c) + h$. Find the values of A, k, c, and h.

A: $|A| = 2.9$

$A = 2.9$ or -2.9

k: $\frac{2\pi}{k} = \frac{2\pi}{5}$ *The period is* $\frac{2\pi}{5}$.

$k = 5$

c: $-\frac{c}{k} = -\frac{\pi}{2}$ *The phase shift is* $-\frac{\pi}{2}$.

$-\frac{c}{5} = -\frac{\pi}{2}$ $k = 5$

$c = \frac{5\pi}{2}$

h: $h = -3$

The possible equations are $y = \pm 2.9 \cos \left(5\theta + \frac{5\pi}{2}\right) - 3$.

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Study Guide

Modeling Real-World Data with Sinusoidal Functions

Example **The table shows the average monthly temperatures for Ann Arbor, Michigan. Write a sinusoidal function that models the average monthly temperatures, using $t = 1$ to represent January. Temperatures are in degrees Fahrenheit (°F).**

Jan.	30°
Feb.	34°
Mar.	45°
Apr.	59°
May	71°
June	80°
July	84°
Aug.	81°
Sept.	74°
Oct.	62°
Nov.	48°
Dec.	35°

These data can be modeled by a function of the form $y = A \sin(kt + c) + h$, where t is the time in months.

First, find A, h, and k.

A: $A = \frac{84 - 30}{2}$ or 27 — *A is half the difference between the greatest temperature and the least temperature.*

h: $h = \frac{84 + 30}{2}$ or 57 — *h is half the sum of the greatest value and the least value.*

k: $\frac{2\pi}{k} = 12$ — *The period is 12.*

$k = \frac{\pi}{6}$

Substitute these values into the general form of the function.

$y = A \sin(kt + c) + h \qquad y = 27 \sin\left(\frac{\pi}{6}t + c\right) + 57$

To compute c, substitute one of the coordinate pairs into the equation.

$$y = 27 \sin\left(\frac{\pi}{6}t + c\right) + 57$$

$$30 = 27 \sin\left[\frac{\pi}{6}(1) + c\right] + 57 \qquad \textit{Use } (t, y) = (1, 30).$$

$$-27 = 27 \sin\left(\frac{\pi}{6} + c\right) \qquad \textit{Subtract 57 from each side.}$$

$$-\frac{27}{27} = \sin\left(\frac{\pi}{6} + c\right) \qquad \textit{Divide each side by 27.}$$

$$\sin^{-1}(-1) = \frac{\pi}{6} + c \qquad \textit{Definition of inverse}$$

$$\sin^{-1}(-1) - \frac{\pi}{6} = c \qquad \textit{Subtract } \frac{\pi}{6} \textit{ from each side.}$$

$$-2.094395102 \approx c \qquad \textit{Use a calculator.}$$

The function $y = 27 \sin\left(\frac{\pi}{6}t - 2.09\right) + 57$ is one model for the average monthly temperature in Ann Arbor, Michigan.

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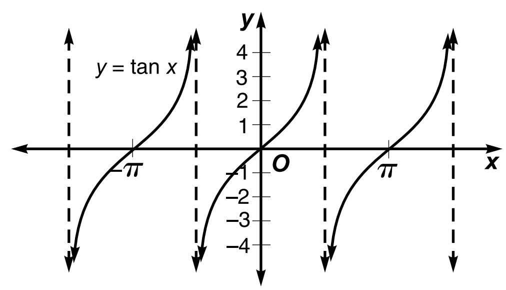
Study Guide

Graphing Other Trigonometric Functions

The period of functions $y = \csc k\theta$ and $y = \sec k\theta$ is $\frac{2\pi}{k}$, where $k > 0$. The period of functions $y = \tan k\theta$ and $y = \cot k\theta$ is $\frac{\pi}{k}$, where $k > 0$. The phase shift and vertical shift work the same way for all trigonometric functions. For example, the phase shift of the function $y = \tan (k\theta + c) + h$ is $-\frac{c}{k}$, and its vertical shift is h.

Example 1 **Graph $y = \tan x$.**

To graph $y = \tan x$, first draw the asymptotes located at $x = \frac{\pi}{2}n$, where n is an odd integer. Then plot the following coordinate pairs and draw the curves.



$\left(-\frac{5\pi}{4}, -1\right)$, $(-\pi, 0)$, $\left(-\frac{3\pi}{4}, 1\right)$, $\left(-\frac{\pi}{4}, -1\right)$,
$(0, 0)$, $\left(\frac{\pi}{4}, 1\right)$, $\left(\frac{3\pi}{4}, -1\right)$, $(\pi, 0)$, $\left(\frac{5\pi}{4}, 1\right)$

Notice that the range values for the interval $-\frac{3\pi}{2} \leq x \leq -\frac{\pi}{2}$ repeat for the intervals $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$.
So, the tangent function is a periodic function with a period of $\frac{\pi}{k}$ or π.

Example 2 **Graph $y = \sec (2\theta + \pi) + 4$.**

Since $k = 2$, the period is $\frac{2\pi}{2}$ or π. Since $c = \pi$, the phase shift is $-\frac{\pi}{2}$. The vertical shift is 4.

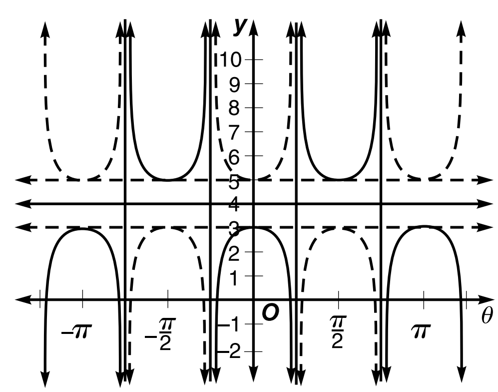
Using this information, follow the steps for graphing a secant function.

Step 1 Draw the midline, which is the graph of $y = 4$.

Step 2 Draw dashed lines parallel to the midline, which are 1 unit above and below $y = 4$.

Step 3 Draw the secant curve with a period of π.

Step 4 Shift the graph $\frac{\pi}{2}$ units to the left.



Study Guide

Trigonometric Inverses and Their Graphs

The inverses of the Sine, Cosine, and Tangent functions are called Arcsine, Arccosine, and Arctangent, respectively. The capital letters are used to represent the functions with restricted domains. The graphs of Arcsine, Arccosine, and Arctangent are defined as follows.

Arcsine Function	Given $y = \text{Sin}\, x$, the inverse Sine function is defined by the equation $y = \text{Sin}^{-1} x$ or $y = \text{Arcsin}\, x$.
Arccosine Function	Given $y = \text{Cos}\, x$, the inverse Cosine function is defined by the equation $y = \text{Cos}^{-1} x$ or $y = \text{Arccos}\, x$.
Arctangent Function	Given $y = \text{Tan}\, x$, the inverse Tangent function is defined by the equation $y = \text{Tan}^{-1} x$ or $y = \text{Arctan}\, x$.

Example 1 **Write the equation for the inverse of $y = \text{Arcsin}\, 2x$. Then graph the function and its inverse.**

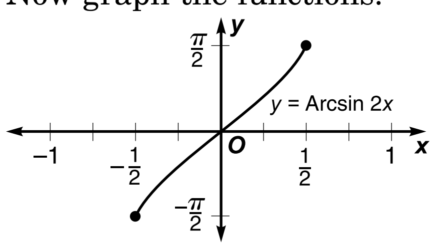
$y = \text{Arcsin}\, 2x$

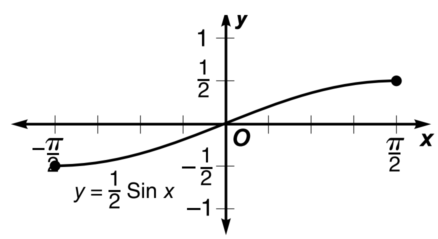
$x = \text{Arcsin}\, 2y$ *Exchange x and y.*

$\text{Sin}\, x = 2y$ *Definition of Arcsin function*

$\frac{1}{2}\text{Sin}\, x = y$ *Divide each side by 2.*

Now graph the functions.





Example 2 **Find each value.**

a. $\text{Arctan}\left(-\frac{\sqrt{3}}{3}\right)$

Let $\theta = \text{Arctan}\left(-\frac{\sqrt{3}}{3}\right)$. *$\text{Arctan}\left(-\frac{\sqrt{3}}{3}\right)$ means that angle whose tan is $-\frac{\sqrt{3}}{3}$.*

$\text{Tan}\, \theta = -\frac{\sqrt{3}}{3}$ *Definition of Arctan function*

$\theta = -\frac{\pi}{6}$

b. $\text{Cos}^{-1}\left(\sin \frac{\pi}{2}\right)$

If $y = \sin \frac{\pi}{2}$, then $y = 1$.

$\text{Cos}^{-1}\left(\sin \frac{\pi}{2}\right) = \text{Cos}^{-1} 1$ *Replace $\sin \frac{\pi}{2}$ with 1.*

$= 0$

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Study Guide

Basic Trigonometric Identities

You can use the **trigonometric identities** to help find the values of trigonometric functions.

Example 1 **If $\sin\theta = \frac{3}{5}$, find $\tan\theta$.**

Use two identities to relate $\sin\theta$ and $\tan\theta$.

$\sin^2\theta + \cos^2\theta = 1$ *Pythagorean identity*

$\left(\frac{3}{5}\right)^2 + \cos^2\theta = 1$ *Substitute $\frac{3}{5}$ for $\sin\theta$.*

$\cos^2\theta = \frac{16}{25}$

$\cos\theta = \pm\sqrt{\frac{16}{25}}$ or $\pm\frac{4}{5}$

Now find $\tan\theta$.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ *Quotient identity*

$\tan\theta = \frac{\frac{3}{5}}{\pm\frac{4}{5}}$

$\tan\theta = \pm\frac{3}{4}$

To determine the sign of a function value, use the **symmetry identities** for sine and cosine. To use these identities with radian measure, replace 180° with π and 360° with 2π.

Case 1:	$\sin(A + 360k°) = \sin A$	$\cos(A + 360k°) = \cos A$
Case 2:	$\sin[A + 180°(2k - 1)] = -\sin A$	$\cos[A + 180°(2k - 1)] = -\cos A$
Case 3:	$\sin(360k° - A) = -\sin A$	$\cos(360k° - A) = \cos A$
Case 4:	$\sin[180°(2k - 1) - A] = \sin A$	$\cos[180°(2k - 1) - A] = -\cos A$

Example 2 Express $\tan\frac{11\pi}{3}$ as a trigonometric function of an angle in Quadrant I.

The sum of $\frac{11\pi}{3}$ and $\frac{\pi}{3}$, which is $\frac{12\pi}{3}$ or 4π, is a multiple of 2π.

$\frac{11\pi}{3} = 2(2\pi) - \frac{\pi}{3}$ *Case 3, with $A = \frac{\pi}{3}$ and $k = 2$*

$\tan\frac{11\pi}{3} = \frac{\sin\frac{11\pi}{3}}{\cos\frac{11\pi}{3}}$ *Quotient identity*

$= \frac{\sin\left[2(2\pi) - \frac{\pi}{3}\right]}{\cos\left[2(2\pi) - \frac{\pi}{3}\right]}$

$= \frac{-\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}$ *Symmetry identities*

$= -\tan\frac{\pi}{3}$ *Quotient identity*

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Study Guide

Verifying Trigonometric Identities

When verifying trigonometric identities, you cannot add or subtract quantities from each side of the identity. An unverified identity is not an equation, so the properties of equality do not apply.

Example 1 **Verify that $\frac{\sec^2 x - 1}{\sec^2 x} = \sin^2 x$ is an identity.**

Since the left side is more complicated, transform it into the expression on the right.

$$\frac{\sec^2 x - 1}{\sec^2 x} \stackrel{?}{=} \sin^2 x$$

$$\frac{(\tan^2 x + 1) - 1}{\sec^2 x} \stackrel{?}{=} \sin^2 x \qquad \textit{sec}^2\ x = \textit{tan}^2\ x + 1$$

$$\frac{\tan^2 x}{\sec^2 x} \stackrel{?}{=} \sin^2 x \qquad \textit{Simplify.}$$

$$\frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}} \stackrel{?}{=} \sin^2 x \qquad \textit{tan}^2\ x = \frac{\textit{sin}^2\ x}{\textit{cos}^2\ x},\ \textit{sec}^2\ x = \frac{1}{\textit{cos}^2\ x}$$

$$\frac{\sin^2 x}{\cos^2 x} \cdot \cos^2 x \stackrel{?}{=} \sin^2 x$$

$$\sin^2 x = \sin^2 x \qquad \textit{Multiply.}$$

The techniques that you use to verify trigonometric identities can also be used to simplify trigonometric equations.

Example 2 **Find a numerical value of one trigonometric function of x if $\cos x \csc x = 3$.**

You can simplify the trigonometric epression on the left side by writing it in terms of sine and cosine.

$$\cos x \csc x = 3$$

$$\cos x \cdot \frac{1}{\sin x} = 3 \qquad \textit{csc}\ x = \frac{1}{\textit{sin}\ x}$$

$$\frac{\cos x}{\sin x} = 3 \qquad \textit{Multiply.}$$

$$\cot x = 3 \qquad \textit{cot}\ x = \frac{\textit{cos}\ x}{\textit{sin}\ x}$$

Therefore, if $\cos x \csc x = 3$, then $\cot x = 3$.

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Study Guide

Sum and Difference Identities

You can use the **sum and difference identities** and the values of the trigonometric functions of common angles to find the values of trigonometric functions of other angles. Notice how the addition and subtraction symbols are related in the sum and difference identities.

Sum and Difference Identities	
Cosine function	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
Sine function	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
Tangent function	$\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Example 1 **Use the sum or difference identity for cosine to find the exact value of cos 375°.**

$375° = 360° + 15°$

$\cos 375° = \cos 15°$ *Symmetry identity, Case 1*

$\cos 15° = \cos(60° - 45°)$ *60° and 45° are two common angles that differ by 15°.*

$\cos 15° = \cos 60° \cos 45° + \sin 60° \sin 45°$ *Difference identity for cosine*

$\cos 15° = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$ or $\frac{\sqrt{2} + \sqrt{6}}{4}$

Example 2 **Find the value of sin $(x + y)$ if $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$, sin $x = \frac{3}{5}$, and sin $y = \frac{12}{37}$.**

In order to use the sum identity for sine, you need to know cos x and cos y. Use a Pythagorean identity to determine the necessary values.

$\sin^2\alpha + \cos^2\alpha = 1 \Rightarrow \cos^2\alpha = 1 - \sin^2\alpha.$ *Pythagorean identity*

Since it is given that the angles are in Quadrant I, the values of sine and cosine are positive. Therefore, $\cos\alpha = \sqrt{1 - \sin^2\alpha}$.

$\cos x = \sqrt{1 - \left(\frac{3}{5}\right)^2}$ $\qquad$ $\cos y = \sqrt{1 - \left(\frac{12}{37}\right)^2}$

$= \sqrt{\frac{16}{25}}$ or $\frac{4}{5}$ $\qquad$ $= \sqrt{\frac{1225}{1369}}$ or $\frac{35}{37}$

Now substitute these values into the sum identity for sine.

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$= \left(\frac{3}{5}\right)\left(\frac{35}{37}\right) + \left(\frac{4}{5}\right)\left(\frac{12}{37}\right)$ or $\frac{153}{185}$

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Study Guide

Double-Angle and Half-Angle Identities

Example 1 **If $\sin\theta = \frac{1}{4}$ and θ has its terminal side in the first quadrant, find the exact value of $\sin 2\theta$.**

To use the double-angle identity for $\sin 2\theta$, we must first find $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(\frac{1}{4}\right)^2 + \cos^2\theta = 1 \quad \textit{sin } \theta = \frac{1}{4}$$
$$\cos^2\theta = \frac{15}{16}$$
$$\cos\theta = \frac{\sqrt{15}}{4}$$

Now find $\sin 2\theta$.

$\sin 2\theta = 2\sin\theta\cos\theta$ — *Double-angle identity for sine*

$= 2\left(\frac{1}{4}\right)\frac{\sqrt{15}}{4}$ — *$\sin\theta = \frac{1}{4}$, $\cos\theta = \frac{\sqrt{15}}{4}$*

$= \frac{\sqrt{15}}{8}$

Example 2 **Use a half-angle identity to find the exact value of $\sin\frac{\pi}{12}$.**

$$\sin\frac{\pi}{12} = \sin\frac{\frac{\pi}{6}}{2}$$

$= \sqrt{\frac{1 - \cos\frac{\pi}{6}}{2}}$ — *Use $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$. Since $\frac{\pi}{12}$ is in Quadrant I, choose the positive sine value.*

$$= \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}}$$
$$= \sqrt{\frac{2 - \sqrt{3}}{4}}$$
$$= \frac{\sqrt{2 - \sqrt{3}}}{2}$$

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Study Guide

Solving Trigonometric Equations

When you solve trigonometric equations for **principal values** of x, x is in the interval $-90° \leq x \leq 90°$ for $\sin x$ and $\tan x$. For $\cos x$, x is in the interval $0° \leq x \leq 180°$. If an equation cannot be solved easily by factoring, try writing the expressions in terms of only one trigonometric function.

Example 1 **Solve $\tan x \cos x - \cos x = 0$ for principal values of x. Express solutions in degrees.**

$$\tan x \cos x - \cos x = 0$$
$$\cos x (\tan x - 1) = 0 \quad \textit{Factor.}$$
$$\cos x = 0 \quad \text{or} \quad \tan x - 1 = 0 \quad \textit{Set each factor equal to 0.}$$
$$x = 90° \qquad \tan x = 1$$
$$x = 45°$$

When $x = 90°$, $\tan x$ is undefined, so the only principal value is 45°.

Example 2 **Solve $2 \tan^2 x - \sec^2 x + 3 = 1 - 2 \tan x$ for $0 \leq x < 2\pi$.**

This equation can be written in terms of $\tan x$ only.

$$2 \tan^2 x - \sec^2 x + 3 = 1 - 2 \tan x$$
$$2 \tan^2 x - (\tan^2 x + 1) + 3 = 1 - 2 \tan x \quad \textit{sec}^2 x = \textit{tan}^2 x + 1$$
$$\tan^2 x + 2 = 1 - 2 \tan x \quad \textit{Simplify.}$$
$$\tan^2 x + 2 \tan x + 1 = 0$$
$$(\tan x + 1)^2 = 0 \quad \textit{Factor.}$$
$$\tan x + 1 = 0 \quad \textit{Take the square root of each side.}$$
$$\tan x = -1$$
$$x = \frac{3\pi}{4} \text{ or } x = \frac{7\pi}{4}$$

When you solve for all values of x, the solution should be represented as $x + 360°k$ or $x + 2\pi k$ for $\sin x$ and $\cos x$ and $x + 180°k$ or $x + \pi k$ for $\tan x$, where k is any integer.

Example 3 **Solve $\sin x + \sqrt{3} = -\sin x$ for all real values of x.**

$$\sin x + \sqrt{3} = -\sin x$$
$$2 \sin x + \sqrt{3} = 0$$
$$2 \sin x = -\sqrt{3}$$
$$\sin x = -\frac{\sqrt{3}}{2}$$
$$x = \frac{4\pi}{3} + 2\pi k \text{ or } x = \frac{5\pi}{3} + 2\pi k, \text{ where } k \text{ is any integer}$$

The solutions are $\frac{4\pi}{3} + 2\pi k$ and $\frac{5\pi}{3} + 2\pi k$.

7-6

Study Guide

Normal Form of a Linear Equation

Normal Form	The normal form of a linear equation is $x \cos \phi + y \sin \phi - p = 0$, where p is the length of the normal from the line to the origin and ϕ is the positive angle formed by the positive x-axis and the normal.

You can write the standard form of a linear equation if you are given the values of ϕ and p.

Example 1 **Write the standard form of the equation of a line for which the length of the normal segment to the origin is 5 and the normal makes an angle of 135° with the positive *x*-axis.**

$$x \cos \phi + y \sin \phi - p = 0 \quad \textit{Normal form}$$
$$x \cos 135° + y \sin 135° - 5 = 0 \quad \phi = 135° \textit{ and } p = 5$$
$$-\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y - 5 = 0$$
$$\sqrt{2}x - \sqrt{2}y + 10 = 0 \quad \textit{Multiply each side by } -2.$$

The standard form of the equation is $\sqrt{2}x - \sqrt{2}y + 10 = 0$.

The standard form of a linear equation, $Ax + By + C = 0$, can be changed to the normal form by dividing each term of the equation by $\pm\sqrt{A^2 + B^2}$. The sign is chosen opposite the sign of C. You can then find the length of the normal, p units, and the angle ϕ.

Example 2 **Write $3x + 4y - 10 = 0$ in normal form. Then find the length of the normal and the angle it makes with the positive *x*-axis.**

Since C is negative, use $\sqrt{A^2 + B^2}$ to determine the normal form.

$\sqrt{A^2 + B^2} = \sqrt{3^2 + 4^2}$ or 5

The normal form is $\frac{3}{5}x + \frac{4}{5}y - \frac{10}{5} = 0$ or $\frac{3}{5}x + \frac{4}{5}y - 2 = 0$.

Therefore, $\cos \phi = \frac{3}{5}$, $\sin \phi = \frac{4}{5}$, and $p = 2$.

Since $\cos \phi$ and $\sin \phi$ are both positive, ϕ must lie in Quadrant I.

$$\tan \phi = \frac{\frac{4}{5}}{\frac{3}{5}} \text{ or } \frac{4}{3} \qquad \tan \phi = \frac{\sin \phi}{\cos \phi}$$
$$\phi \approx 53°$$

The normal segment has length 2 units and makes an angle of 53° with the positive x-axis.

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Study Guide

Distance from a Point to a Line

The distance from a point at (x_1, y_1) to a line with equation $Ax + By + C = 0$ can be determined by using the formula $d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$. The sign of the radical is chosen opposite the sign of C.

Example 1 **Find the distance between $P(3, 4)$ and the line with equation $4x + 2y = 10$.**

First, rewrite the equation of the line in standard form.

$$4x + 2y - 10 = 0$$

Then, use the formula for the distance from a point to a line.

$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$

$d = \frac{4(3) + 2(4) - 10}{\pm\sqrt{4^2 + 2^2}}$ *$A = 4, B = 2, C = -10, x_1 = 3, y_1 = 4$*

$d = \frac{10}{2\sqrt{5}}$ or $\sqrt{5}$ *Since C is negative, use $+\sqrt{A^2 + B^2}$.*

$d \approx 2.24$ units

Therefore, P is approximately 2.24 units from the line with equation $4x + 2y = 10$. Since d is positive, P is on the opposite side of the line from the origin.

You can also use the formula to find the distance between two parallel lines. To do this, choose any point on one of the lines and use the formula to find the distance from that point to the other line.

Example 2 **Find the distance between the lines with equations $2x - 2y = 5$ and $y = x - 1$.**

Since $y = x - 1$ is in slope-intercept form, you can see that it passes through the point at $(0, -1)$. Use this point to find the distance to the other line.

The standard form of the other equation is $2x - 2y - 5 = 0$.

$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$

$d = \frac{2(0) - 2(-1) - 5}{\pm\sqrt{2^2 + (-2)^2}}$ *$A = 2, B = -2, C = -5, x_1 = 0, y_1 = -1$*

$d = -\frac{3}{2\sqrt{2}}$ or $-\frac{3\sqrt{2}}{4}$ *Since C is negative, use $+\sqrt{A^2 + B^2}$.*

≈ -1.06

The distance between the lines is about 1.06 units.

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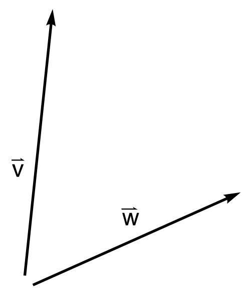
8-1

Study Guide

Geometric Vectors

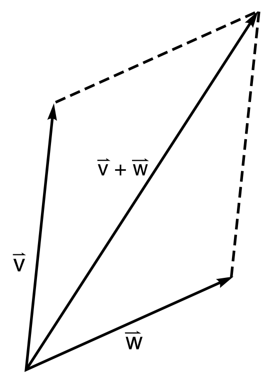
The **magnitude** of a **vector** is the length of a directed line segment. The **direction** of the vector is the directed angle between the positive x-axis and the vector. When adding or subtracting vectors, use either the parallelogram or the triangle method to find the **resultant**.

Example 1 **Use the parallelogram method to find the sum of $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$.**



Copy $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, placing the initial points together.
Form a parallelogram that has $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ as two of its sides.
Draw dashed lines to represent the other two sides.

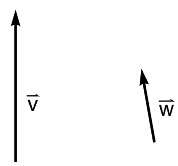
The resultant is the vector from the vertex of $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ to the opposite vertex of the parallelogram.



Use a ruler and protractor to measure the magnitude and direction of the resultant.

The magnitude is 6 centimeters, and the direction is 40°.

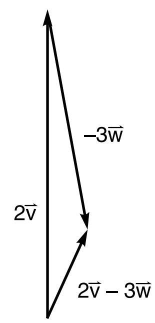
Example 2 **Use the triangle method to find $2\vec{\mathbf{v}} - 3\vec{\mathbf{w}}$.**



$2\vec{\mathbf{v}} - 3\vec{\mathbf{w}} = 2\vec{\mathbf{v}} + (-3\vec{\mathbf{w}})$

Draw a vector that is twice the magnitude of $\vec{\mathbf{v}}$ to represent $2\vec{\mathbf{v}}$. Then draw a vector with the opposite direction to $\vec{\mathbf{w}}$ and three times its magnitude to represent $-3\vec{\mathbf{w}}$. Place the initial point of $-3\vec{\mathbf{w}}$ on the terminal point of $2\vec{\mathbf{v}}$. *Tip-to-tail method.*

Draw the resultant from the initial point of the first vector to the terminal point of the second vector. The resultant is $2\vec{\mathbf{v}} - 3\vec{\mathbf{w}}$.



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8-2

Study Guide

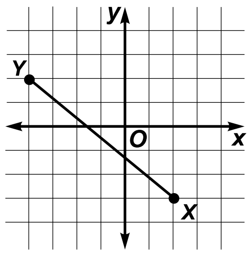
Algebraic Vectors

Vectors can be represented algebraically using ordered pairs of real numbers.

Example 1 **Write the ordered pair that represents the vector from $X(2, -3)$ to $Y(-4, 2)$. Then find the magnitude of $\overrightarrow{XY}$.**

First represent $\overrightarrow{XY}$ as an ordered pair.

$$\begin{aligned}\overrightarrow{XY} &= \langle x_2 - x_1, y_2 - y_1\rangle \\ &= \langle -4 - 2, 2 - (-3)\rangle \\ &= \langle -6, 5\rangle\end{aligned}$$



Then determine the magnitude of $\overrightarrow{XY}$.

$$\begin{aligned}|\overrightarrow{XY}| &= \sqrt{(x_2 - x_1)^2, (y_2 - y_1)^2} \\ &= \sqrt{(-4 - 2)^2 + [2 - (-3)]^2} \\ &= \sqrt{(-6)^2 + 5^2} \\ &= \sqrt{61}\end{aligned}$$

$\overrightarrow{XY}$ is represented by the ordered pair $\langle -6, 5\rangle$ and has a magnitude of $\sqrt{61}$ units.

Example 2 **Let $\vec{\mathbf{s}} = \langle 4, 2\rangle$ and $\vec{\mathbf{t}} = \langle -1, 3\rangle$. Find each of the following.**

a. $\vec{\mathbf{s}} + \vec{\mathbf{t}}$

$$\begin{aligned}\vec{\mathbf{s}} + \vec{\mathbf{t}} &= \langle 4, 2\rangle + \langle -1, 3\rangle \\ &= \langle 4 + (-1), 2 + 3\rangle \\ &= \langle 3, 5\rangle\end{aligned}$$

b. $\vec{\mathbf{s}} - \vec{\mathbf{t}}$

$$\begin{aligned}\vec{\mathbf{s}} - \vec{\mathbf{t}} &= \langle 4, 2\rangle - \langle -1, 3\rangle \\ &= \langle 4 - (-1), 2 - 3\rangle \\ &= \langle 5, -1\rangle\end{aligned}$$

c. $4\vec{\mathbf{s}}$

$$\begin{aligned}4\vec{\mathbf{s}} &= 4\langle 4, 2\rangle \\ &= \langle 4 \cdot 4, 4 \cdot 2\rangle \\ &= \langle 16, 8\rangle\end{aligned}$$

d. $3\vec{\mathbf{s}} + \vec{\mathbf{t}}$

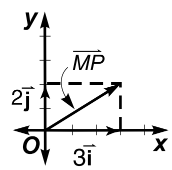
$$\begin{aligned}3\vec{\mathbf{s}} + \vec{\mathbf{t}} &= 3\langle 4, 2\rangle + \langle -1, 3\rangle \\ &= \langle 12, 6\rangle + \langle -1, 3\rangle \\ &= \langle 11, 9\rangle\end{aligned}$$

A unit vector in the direction of the positive x-axis is represented by $\vec{\mathbf{i}}$, and a unit vector in the direction of the positive y-axis is represented by $\vec{\mathbf{j}}$. Vectors represented as ordered pairs can be written as the sum of unit vectors.

Example 3 **Write $\overrightarrow{MP}$ as the sum of unit vectors for $M(2, 2)$ and $P(5, 4)$.**

First write $\overrightarrow{MP}$ as an ordered pair.

$$\begin{aligned}\overrightarrow{MP} &= \langle 5 - 2, 4 - 2\rangle \\ &= \langle 3, 2\rangle\end{aligned}$$



Then write $\overrightarrow{MP}$ as the sum of unit vectors.

$\overrightarrow{MP} = 3\vec{\mathbf{i}} + 2\vec{\mathbf{j}}$

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8-3

Study Guide

Vectors in Three-Dimensional Space

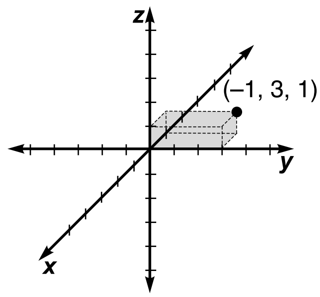
Ordered triples, like ordered pairs, can be used to represent vectors. Operations on vectors respresented by ordered triples are similar to those on vectors represented by ordered pairs. For example, an extension of the formula for the distance between two points in a plane allows us to find the distance between two points in space.

Example 1 **Locate the point at (−1, 3, 1).**

Locate −1 on the x-axis, 3 on the y-axis, and 1 on the z-axis.

Now draw broken lines for parallelograms to represent the three planes.

The planes intersect at (−1, 3, 1).



Example 2 **Write the ordered triple that represents the vector from $X(-4, 5, 6)$ to $Y(-2, 6, 3)$. Then find the magnitude of $\overrightarrow{XY}$.**

$$\begin{aligned}\overrightarrow{XY} &= (-2, 6, 3) - (-4, 5, 6)\\ &= \langle -2 - (-4), 6 - 5, 3 - 6\rangle\\ &= \langle 2, 1, -3\rangle\end{aligned}$$

$$\begin{aligned}|\overrightarrow{XY}| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\\ &= \sqrt{[-2 - (-4)]2 + (6 - 5)2 + (3 - 6)2}\\ &= \sqrt{(2)^2 + (1)^2 + (-3)^2}\\ &= \sqrt{14} \text{ or } 3.7\end{aligned}$$

Example 3 **Find an ordered triple that represents $2\vec{s} + 3\vec{t}$ if $\vec{s} = \langle 5, -1, 2\rangle$ and $\vec{t} = \langle 4, 3, -2\rangle$.**

$$\begin{aligned}2\vec{\mathbf{s}} + 3\vec{\mathbf{t}} &= 2\langle 5, -1, 2\rangle + 3\langle 4, 3, -2\rangle\\ &= \langle 10, -2, 4\rangle + \langle 12, 9, -6\rangle\\ &= \langle 22, 7, -2\rangle\end{aligned}$$

Example 4 **Write $\overrightarrow{AB}$ as the sum of unit vectors for $A(5, -2, 3)$ and $B(-4, 2, 1)$.**

First express $\overrightarrow{AB}$ as an ordered triple. Then write the sum of the unit vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$.

$$\begin{aligned}\overrightarrow{AB} &= (-4, 2, 1) - (5, -2, 3)\\ &= \langle -4 - 5, 2 - (-2), 1 - 3\rangle\\ &= \langle -9, 4, -2\rangle\\ &= -9\vec{\mathbf{i}} + 4\vec{\mathbf{j}} - 2\vec{\mathbf{k}}\end{aligned}$$

8-4

Study Guide

Perpendicular Vectors

Two vectors are perpendicular if and only if their **inner product** is zero.

Example 1 **Find each inner product if $\vec{\mathbf{u}} = \langle 5, 1\rangle$, $\vec{\mathbf{v}} = \langle -3, 15\rangle$, and $\vec{\mathbf{w}} = \langle 2, -1\rangle$. Is either pair of vectors perpendicular?**

a. $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$

$$\begin{aligned}\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= 5(-3) + 1(15)\\ &= -15 + 15\\ &= 0\end{aligned}$$

$\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are perpendicular.

b. $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$

$$\begin{aligned}\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} &= -3(2) + 15(-1)\\ &= -6 + (-15)\\ &= -21\end{aligned}$$

$\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are not perpendicular.

Example 2 **Find the inner product of $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$ if $\vec{\mathbf{r}} = \langle 3, -1, 0\rangle$ and $\vec{\mathbf{s}} = \langle 2, 6, 4\rangle$. Are $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$ perpendicular?**

$$\begin{aligned}\vec{\mathbf{r}} \cdot \vec{\mathbf{s}} &= (3)(2) + (-1)(6) + (0)(4)\\ &= 6 + (-6) + 0\\ &= 0\end{aligned}$$

$\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$ are perpendicular since their inner product is zero.

Unlike the inner product, the **cross product** of two vectors is a vector. This vector does not lie in the plane of the given vectors but is perpendicular to the plane containing the two vectors.

Example 3 **Find the cross product of $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ if $\vec{\mathbf{v}} = \langle 0, 4, 1\rangle$ and $\vec{\mathbf{w}} = \langle 0, 1, 3\rangle$. Verify that the resulting vector is perpendicular to $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$.**

$$\vec{\mathbf{v}} \times \vec{\mathbf{w}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 0 & 4 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix}\vec{\mathbf{i}} - \begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix}\vec{\mathbf{j}} + \begin{vmatrix} 0 & 4 \\ 0 & 1 \end{vmatrix}\vec{\mathbf{k}} \quad \textit{Expand by minors.}$$

$$= 11\vec{\mathbf{i}} - 0\vec{\mathbf{j}} + 0\vec{\mathbf{k}}$$

$$= 11\vec{\mathbf{i}} \text{ or } \langle 11, 0, 0\rangle$$

Find the inner products.

$\langle 11, 0, 0\rangle \cdot \langle 0, 4, 1\rangle$
$11(0) + 0(4) + 0(1) = 0$

$\langle 11, 0, 0\rangle \cdot \langle 0, 1, 3\rangle$
$11(0) + 0(1) + 0(3) = 0$

Since the inner products are zero, the cross product $\vec{\mathbf{v}} \times \vec{\mathbf{w}}$ is perpendicular to both $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$.

8-5

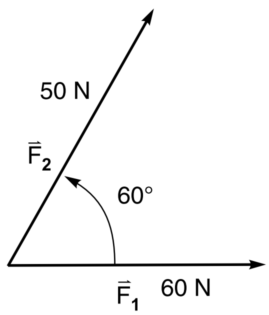
Study Guide

Applications with Vectors

Vectors can be used to represent any quantity that has direction and magnitude, such as force, velocity, and weight.

Example **Suppose Jamal and Mike pull on the ends of a rope tied to a dinghy. Jamal pulls with a force of 60 newtons and Mike pulls with a force of 50 newtons. The angle formed when Jamal and Mike pull on the rope is 60°.**

a. Draw a labeled diagram that represents the forces.



Let $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ represent the two forces.

b. Determine the magnitude of the resultant force.

First find the horizontal (x) and vertical (y) components of each force.

Given that we place $\vec{\mathbf{F}}_1$ on the x-axis, the unit vector is $1\vec{\mathbf{i}} + 0\vec{\mathbf{j}}$.

Therefore, the x- and y-components of $\vec{\mathbf{F}}_1$ are $60\vec{\mathbf{i}} + 0\vec{\mathbf{j}}$.

$\vec{\mathbf{F}}_2 = x\vec{\mathbf{i}} + y\vec{\mathbf{j}}$

$$\cos 60° = \frac{x}{50} \qquad\qquad \sin 60° = \frac{y}{50}$$
$$x = 50 \cos 60° \qquad\qquad y = 50 \sin 60°$$
$$= 25 \qquad\qquad \approx 43.3$$

Thus, $\vec{\mathbf{F}}_2 = 25\vec{\mathbf{i}} + 43.3\vec{\mathbf{j}}$.

Then add the unit components.

$$(60\vec{\mathbf{i}} + 0\vec{\mathbf{j}}) + (25\vec{\mathbf{i}} + 43.3\vec{\mathbf{j}}) = 85\vec{\mathbf{i}} + 43.3\vec{\mathbf{j}}$$

$$\vec{\mathbf{F}} \approx \sqrt{85^2 + 43.3^2}$$
$$\approx \sqrt{9099.89}$$
$$\approx 95.39$$

The magnitude of the resultant force is 95.39 newtons.

c. Determine the direction of the resultant force.

$\tan \theta = \frac{43.3}{85}$ *Use the tangent ratio.*

$$\theta = \tan^{-1} \frac{43.3}{85}$$
$$\theta \approx 27°$$

The direction of the resultant force is 27° with respect to the vector on the x-axis.

8-6

Study Guide

Vectors and Parametric Equations

Vector equations and **parametric equations** allow us to model movement.

Example 1 **Write a vector equation describing a line passing through $P_1(8, 4)$ and parallel to $\vec{\mathbf{a}} = \langle 6, -1\rangle$. Then write parametric equations of the line.**

Let the line ℓ through $P_1(8, 4)$ be parallel to $\vec{\mathbf{a}}$. For any point $P_2(x, y)$ on ℓ, $\overrightarrow{P_1P_2}\langle x - 8, y - 4\rangle$. Since $\overrightarrow{P_1P_2}$ is on ℓ and is parallel to $\vec{\mathbf{a}}$, $\overrightarrow{P_1P_2} = t\vec{\mathbf{a}}$, for some value t. By substitution, we have $\langle x - 8, y - 4\rangle = t\langle 6, -1\rangle$.

Therefore, the equation $\langle x - 8, y - 4\rangle = t\langle 6, -1\rangle$ is a vector equation describing all of the points (x, y) on ℓ parallel to $\vec{\mathbf{a}}$ through $P_1(8, 4)$.

Use the general form of the parametric equations of a line with $\langle a_1, a_2\rangle = \langle 6, -1\rangle$ and $\langle x_1, y_1\rangle = \langle 8, 4\rangle$.

$$\begin{aligned} x &= x_1 + ta_1 & \qquad y &= y_1 + ta_2 \\ x &= 8 + t(6) & y &= 4 + t(-1) \\ x &= 8 + 6t & y &= 4 - t \end{aligned}$$

Parametric equations for the line are $x = 8 + 6t$ and $y = 4 - t$.

Example 2 **Write an equation in slope-intercept form of the line whose parametric equations are $x = -3 + 4t$ and $y = 3 + 4t$.**

Solve each parametric equation for t.

$$\begin{aligned} x &= -3 + 4t & \qquad y &= 3 + 4t \\ x + 3 &= 4t & y - 3 &= 4t \\ \frac{x + 3}{4} &= t & \frac{y - 3}{4} &= t \end{aligned}$$

Use substitution to write an equation for the line without the variable t.

$$\begin{aligned} \frac{x + 3}{4} &= \frac{y - 3}{4} && \textit{Substitute.} \\ (x + 3)(4) &= 4(y - 3) && \textit{Cross multiply.} \\ 4x + 12 &= 4y - 12 && \textit{Simplify.} \\ y &= x + 6 && \textit{Solve for } y. \end{aligned}$$

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Study Guide

Modeling Motion Using Parametric Equations

We can use the horizontal and vertical components of a projectile to find parametric equations that represent the path of the projectile.

Example 1 **Find the initial horizontal and vertical velocities of a soccer ball kicked with an initial velocity of 33 feet per second at an angle of 29° with the ground.**

$|\vec{\mathbf{v}}_x| = |\vec{\mathbf{v}}| \cos\theta$ $\qquad$ $|\vec{\mathbf{v}}_y| = |\vec{\mathbf{v}}| \sin\theta$

$|\vec{\mathbf{v}}_x| = 33 \cos 29°$ $\qquad$ $|\vec{\mathbf{v}}_y| = 33 \sin 29°$

$|\vec{\mathbf{v}}_x| \approx 29$ $\qquad$ $|\vec{\mathbf{v}}_y| \approx 16$

The initial horizontal velocity is about 29 feet per second and the initial vertical velocity is about 16 feet per second.

The path of a projectile launched from the ground may be described by the parametric equations $x = t|\vec{\mathbf{v}}| \cos\theta$ for horizontal distance and $y = t|\vec{\mathbf{v}}| \sin\theta - \frac{1}{2}gt^2$ for vertical distance, where t is time and g is acceleration due to gravity. Use $g \approx 9.8$ m/s² or 32 ft/s².

Example 2 **A rock is tossed at an initial velocity of 50 meters per second at an angle of 8° with the ground. After 0.8 second, how far has the rock traveled horizontally and vertically?**

First write the position of the rock as a pair of parametric equations defining the postition of the rock for any time t in seconds.

$x = t|\vec{\mathbf{v}}| \cos\theta$ $\qquad$ $y = t|\vec{\mathbf{v}}| \sin\theta - \frac{1}{2}gt^2$

$x = t(50) \cos 8°$ $\qquad$ $y = t(50) \sin 8° - \frac{1}{2}(9.8)t^2$ $\qquad$ $|\vec{v}| = 50$ *m/s*

$x = 50t \cos 8°$ $\qquad$ $y = 50t \sin 8° - 4.9gt^2$

Then find x and y when $t = 0.8$ second.

$x = 50(0.8) \cos 8°$ $\qquad$ $y = 50(0.8) \sin 8° - 4.9(0.8)^2$

≈ 39.61 $\qquad$ ≈ 2.43

After 0.8 second, the rock has traveled about 39.61 meters horizontally and is about 2.43 meters above the ground.

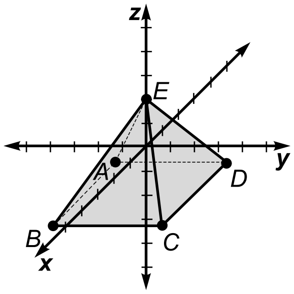
8-8

Study Guide

Transformation Matrices in Three-Dimensional Space

Example 1 **Find the coordinates of the vertices of the pyramid and represent them as a vertex matrix.**

$A(-2, -2, -2)$
$B(2, -2, -2)$
$C(2, 2, -2)$
$D(-2, 2, -2)$
$E(0, 0, 2)$



The vertex matrix for the pyramid is $\begin{array}{c} \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} -2 & 2 & 2 & -2 & 0 \\ -2 & -2 & 2 & 2 & 0 \\ -2 & -2 & -2 & -2 & 2 \end{bmatrix} \end{array}$.

Example 2 **Let M represent the vertex matrix of the pyramid in Example 1.**

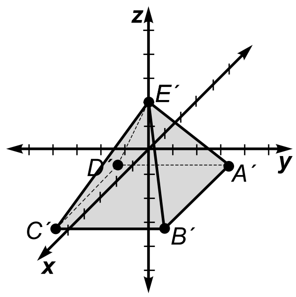
a. Find TM if $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

b. Graph the resulting image and describe the transformation represented by matrix T.

a. First find TM.

$$TM = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 2 & 2 & -2 & 0 \\ -2 & -2 & 2 & 2 & 0 \\ -2 & -2 & -2 & -2 & 2 \end{bmatrix} = \begin{array}{c} \begin{array}{ccccc} A' & B' & C' & D' & E' \end{array} \\ \begin{bmatrix} -2 & 2 & 2 & -2 & 0 \\ 2 & 2 & -2 & -2 & 0 \\ -2 & -2 & -2 & -2 & 2 \end{bmatrix} \end{array}$$

b. Then graph the points represented by the resulting matrix.



The transformation matrix reflects the image of the pyramid over the xz-plane.

9-1

Study Guide

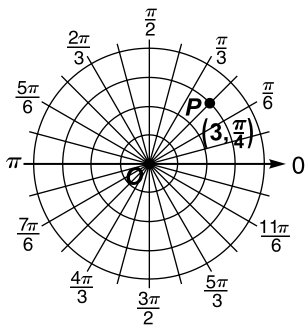
Polar Coordinates

A **polar coordinate system** uses distances and angles to record the position of a point. The location of a point P can be identified by polar coordinates in the form (r, θ), where $|r|$ is the distance from the **pole**, or origin, to point P and θ is the measure of the angle formed by the ray from the pole to point P and the **polar axis.**

Example 1 **Graph each point.**

a. $\boldsymbol{P\left(3, \frac{\pi}{4}\right)}$

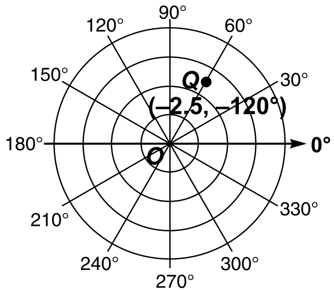
Sketch the terminal side of an angle measuring $\frac{\pi}{4}$ radians in standard position.



Since r is positive ($r = 3$), find the point on the terminal side of the angle that is 3 units from the pole. *Notice point P is on the third circle from the pole.*

b. $\boldsymbol{Q(-2.5, -120°)}$

Negative angles are measured clockwise. Sketch the terminal side of an angle of $-120°$ in standard position.



Since r is negative, extend the terminal side of the angle in the opposite direction. Find the point Q that is 2.5 units from the pole along this extended ray.

Example 2 **Find the distance between $\boldsymbol{P_1(3, 70°)}$ and $\boldsymbol{P_2(5, 120°)}$.**

$$P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}$$
$$= \sqrt{3^2 + 5^2 - 2(3)(5)\cos(120° - 70°)}$$
$$\approx 3.84$$

Study Guide

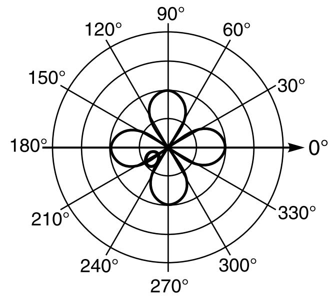
Graphs of Polar Equations

A **polar graph** is the set of all points whose coordinates (r, θ) satisfy a given polar equation. The position and shape of polar graphs can be altered by multiplying the function by a number or by adding to the function. You can also alter the graph by multiplying θ by a number or by adding to it.

Example 1 **Graph the polar equation $r = 2 \cos 2\theta$.**

Make a table of values. Graph the ordered pairs and connect them with a smooth curve.

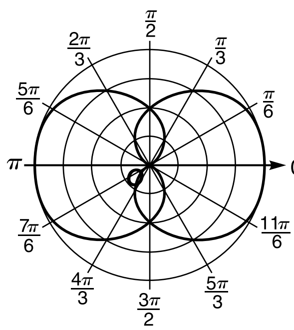
θ	$2 \cos 2\theta$	(r, θ)
0°	2	(2, 0°)
30°	1	(1, 30°)
45°	0	(0, 45°)
60°	−1	(−1, 60°)
90°	−2	(−2, 90°)
120°	−1	(−1, 120°)
135°	0	(0, 135°)
150°	1	(1, 150°)
180°	2	(2, 180°)
210°	1	(1, 210°)
225°	0	(0, 225°)
240°	−1	(−1, 240°)
270°	−2	(−2, 270°)
300°	−1	(−1, 300°)
315°	0	(0, 315°)
330°	1	(1, 330°)



This type of curve is called a *rose*. Notice that the farthest points are 2 units from the pole and the rose has 4 petals.

Example 2 **Graph the system of polar equations. Solve the system using algebra and trigonometry, and compare the solutions to those on your graph.**

$r = 2 + 2 \cos \theta$
$r = 2 - 2 \cos \theta$



To solve the system of equations, substitute $2 + 2 \cos \theta$ for r in the second equation.

$$2 + 2 \cos \theta = 2 - 2 \cos \theta$$
$$\cos \theta = 0$$
$$\theta = \frac{\pi}{2} \text{ or } \theta = \frac{3\pi}{2}$$

Substituting each angle into either of the original equations gives $r = 2$. The solutions of the system are therefore $\left(2, \frac{\pi}{2}\right)$ and $\left(2, \frac{3\pi}{2}\right)$. Tracing on the curves shows that these solutions correspond with two of the intersection points on the graph. The curves also intersect at the pole.

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Study Guide

Polar and Rectangular Coordinates

Use the conversion formulas in the following examples to convert coordinates and equations from one coordinate system to the other.

Example 1 **Find the rectangular coordinates of each point.**

a. $P\left(3, \frac{3\pi}{4}\right)$

For $P\left(3, \frac{3\pi}{4}\right)$, $r = 3$ and $\theta = \frac{3\pi}{4}$. Use the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} x &= r\cos\theta \\ &= 3\cos\frac{3\pi}{4} \\ &= 3\left(-\frac{\sqrt{2}}{2}\right) \\ &\text{or } -\frac{3\sqrt{2}}{2} \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= 3\sin\frac{3\pi}{4} \\ &= 3\left(\frac{\sqrt{2}}{2}\right) \\ &\text{or } \frac{3\sqrt{2}}{2} \end{aligned}$$

The rectangular coordinates of P are $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$, or $(-2.12, 2.12)$ to the nearest hundredth.

b. $Q(20, -60°)$

For $Q(20, -60°)$, $r = 20$ and $\theta = -60°$.

$$\begin{aligned} x &= r\cos\theta \\ &= 20\cos(-60°) \\ &= 20(0.5) \\ &= 10 \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= 20\sin(-60°) \\ &= 20\left(-\frac{\sqrt{3}}{2}\right) \\ &= -10\sqrt{3} \end{aligned}$$

The rectangular coordinates of Q are $(10, -10\sqrt{3})$, or approximately $(10, -17.32)$

Example 2 **Find the polar coordinates of $R(5, -9)$.**

For $R(5, -9)$, $x = 5$ and $y = -9$.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{5^2 + (-9)^2} \\ &= \sqrt{106} \text{ or about } 10.30 \end{aligned} \qquad \begin{aligned} \theta &= \text{Arctan}\,\frac{y}{x} \quad x > 0 \\ &= \text{Arctan}\,\frac{-9}{5} \\ &\approx -1.06 \end{aligned}$$

To obtain an angle between 0 and 2π you can add 2π to the θ-value. This results in $\theta = 5.22$.

The polar coordinates of R are approximately $(10.30, 5.22)$.

Example 3 **Write the polar equation $r = 5\cos\theta$ in rectangular form.**

$$\begin{aligned} r &= 5\cos\theta \\ r^2 &= 5r\cos\theta && \textit{Multiply each side by } r. \\ x^2 + y^2 &= 5x && r^2 = x^2 + y^2 \textit{ and } r\cos\theta = x \end{aligned}$$

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Study Guide

Polar Form of a Linear Equation

Example 1 **Write the equation $x + 3y = 6$ in polar form.**

The standard form of the equation is $x + 3y - 6 = 0$. To find the values of p and ϕ, write the equation in normal form. To convert to normal form, find the value of $\pm\sqrt{A^2 + B^2}$.

$\pm\sqrt{A^2 + B^2} = \pm\sqrt{1^2 + 3^2}$ or $\pm\sqrt{10}$

Since C is negative, use $+\sqrt{10}$.

The normal form of the equation is
$\frac{1}{\sqrt{10}}x + \frac{3}{\sqrt{10}}y - \frac{6}{\sqrt{10}} = 0$ or $\frac{\sqrt{10}}{10}x + \frac{3\sqrt{10}}{10}y - \frac{3\sqrt{10}}{5} = 0$.

Using the normal form $x \cos \phi + y \sin \phi - p = 0$, we can see that $p = \frac{6}{\sqrt{10}}$ or $\frac{3\sqrt{10}}{5}$. Since $\cos \phi$ and $\sin \phi$ are both positive, the normal lies in Quadrant I.

$\tan \phi = \frac{\sin \phi}{\cos \phi}$

$\tan \phi = 3$ $\frac{3}{\sqrt{10}} \div \frac{1}{\sqrt{10}} = 3$

$\phi \approx 1.25$ *Use the Arctangent function.*

Substitute the values for p and ϕ into the polar form.

$p = r \cos(\theta - \phi)$

$\frac{3\sqrt{10}}{5} = r \cos(\theta - 1.25)$ *Polar form of $x + 3y = 6$*

Example 2 **Write $3 = r \cos(\theta - 30°)$ in rectangular form.**

$3 = r \cos(\theta - 30°)$

$3 = r(\cos \theta \cos 30° + \sin \theta \sin 30°)$ *Difference identity for cosine*

$3 = r\left(\frac{\sqrt{3}}{2} \cos \theta + \frac{1}{2} \sin \theta\right)$ *$\cos 30° = \frac{\sqrt{3}}{2}$, $\sin 30° = \frac{1}{2}$*

$3 = \frac{\sqrt{3}}{2} r \cos \theta + \frac{1}{2} r \sin \theta$ *Distributive property*

$3 = \frac{\sqrt{3}}{2} x + \frac{1}{2} y$ *$r \cos \theta = x$, $r \sin \theta = y$*

$6 = \sqrt{3}x + y$ *Multiply each side by 2.*

$0 = \sqrt{3}x + y - 6$ *Subtract 6 from each side.*

The rectangular form of $3 = r\cos(\theta - 30°)$ is $\sqrt{3}x + y - 6 = 0$.

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Study Guide

Simplifying Complex Numbers

Add and subtract complex numbers by performing the chosen operation on both the **real** and **imaginary parts.** Find the product of two or more complex numbers by using the same procedures used to multiply binomials. To simplify the quotient of two complex numbers, multiply the numerator and denominator by the complex conjugate of the denominator.

To find the value of i^n, let R be the remainder when n is divided by 4.	
if R = 0	$i^n = 1$
if R = 1	$i^n = i$
if R = 2	$i^n = -1$
if R = 3	$i^n = -i$

Powers of i	
$i^1 = i$	$i^2 = -1$
$i^3 = i^2 \cdot i = -i$	$i^4 = (i^2)^2 = 1$
$i^5 = i^4 \cdot i = i$	$i^6 = i^4 \cdot i^2 = -1$
$i^7 = i^4 \cdot i^3 = -i$	$i^8 = (i^2)^4 = 1$

Example 1 **Simplify each power of i.**

a. i^{30}

Method 1

$30 \div 4 = 7 \text{ R } 2$

If R = 2, $i^n = -1$.

$i^{30} = -1$

Method 2

$$\begin{aligned} i^{30} &= (i^4)^7 \cdot i^2 \\ &= (1)^7 \cdot i^2 \\ &= -1 \end{aligned}$$

b. i^{-11}

Method 1

$-11 \div 4 = -3 \text{ R } 1$

If R = 1, $i^n = i$.

$i^{-11} = i$

Method 2

$$\begin{aligned} i^{-11} &= (i^4)^{-3} \cdot i^1 \\ &= (1)^{-3} \cdot i^1 \\ &= i \end{aligned}$$

Example 2 **Simplify each expression.**

a. $(3 + 2i) + (5 - 3i)$

$$\begin{aligned} &(3 + 2i) + (5 - 3i) \\ &\quad = (3 + 5) + (2i - 3i) \\ &\quad = 8 - i \end{aligned}$$

b. $(8 - 4i) - (9 - 7i)$

$$\begin{aligned} &(8 - 4i) - (9 - 7i) \\ &\quad = 8 - 4i - 9 + 7i \\ &\quad = -1 + 3i \end{aligned}$$

Example 3 **Simplify $(4 - 2i)(5 - 3i)$.**

$$\begin{aligned} (4 - 2i)(5 - 3i) &= 5(4 - 2i) - 3i(4 - 2i) && \textit{Distributive property} \\ &= 20 - 10i - 12i + 6i^2 && \textit{Distributive property} \\ &= 20 - 10i - 12i + 6(-1) && i^2 = -1 \\ &= 14 - 22i \end{aligned}$$

Example 4 **Simplify $(4 - 5i) \div (2 + i)$.**

$$\begin{aligned} (4 - 5i) \div (2 + i) &= \frac{4 - 5i}{2 + i} \\ &= \frac{4 - 5i}{2 + i} \cdot \frac{2 - i}{2 - i} && 2 - i \textit{ is the conjugate of } 2 + i. \\ &= \frac{8 - 10i - 4i + 5i^2}{4 - i^2} \\ &= \frac{8 - 14i + 5(-1)}{4 - (-1)} && i^2 = -1 \\ &= \frac{3 - 14i}{5} \\ &= \frac{3}{5} - \frac{14}{5}i && \textit{Write the answer in the form } a + bi. \end{aligned}$$

9-6

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Study Guide

The Complex Plane and Polar Form of Complex Numbers

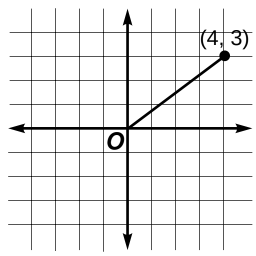
In the **complex plane,** the real axis is horizontal and the imaginary axis is vertical. The **absolute value** of a complex number is its distance from zero in the complex plane.

The polar form of the complex number $a + b\boldsymbol{i}$ is $r(\cos \theta + \boldsymbol{i} \sin \theta)$, which is often abbreviated as $r \text{ cis } \theta$. In polar form, r represents the absolute value, or **modulus,** of the complex number. The angle θ is called the **amplitude** or **argument** of the complex number.

Example 1 **Graph each number in the complex plane and find its absolute value.**

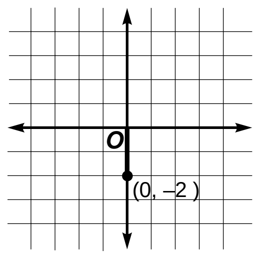
a. $\boldsymbol{z = 4 + 3i}$

$$|z| = \sqrt{4^2 + 3^2}$$
$$= 5$$



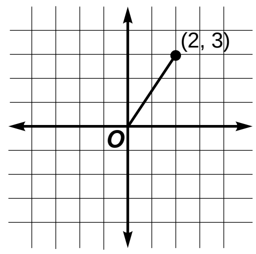
b. $\boldsymbol{z = -2i}$

$$z = 0 - 2\boldsymbol{i}$$
$$|z| = \sqrt{0^2 + (-2)^2}$$
$$= 2$$



Example 2 **Express the complex number $\boldsymbol{2 + 3i}$ in polar form.**

First plot the number in the complex plane.



Then find the modulus.

$r = \sqrt{2^2 + 3^2}$ or $\sqrt{13}$

Now find the amplitude. Notice that θ is in Quadrant I.

$\theta = \text{Arctan } \frac{3}{2}$ $\qquad$ *$\theta = Arctan\ \frac{b}{a}$ if $a > 0$*

≈ 0.98

Therefore, $2 + 3\boldsymbol{i} \approx \sqrt{13}(\cos 0.98 + \boldsymbol{i} \sin 0.98)$ or $\sqrt{13} \text{ cis } 0.98$.

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Study Guide

Products and Quotients of Complex Numbers in Polar Form

Example 1 **Find the product $2\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) \cdot 4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$. Then express the product in rectangular form.**

Find the modulus and amplitude of the product.

$$r = r_1 r_2 = 2(4) = 8 \qquad \theta = \theta_1 + \theta_2 = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$$

The product is $8\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$.

Now find the rectangular form of the product.

$$8\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right) = 8\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \qquad \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2},\ \sin \frac{5\pi}{6} = \frac{1}{2}$$
$$= -4\sqrt{3} + 4i$$

The rectangular form of the product is $-4\sqrt{3} + 4i$.

Example 2 **Find the quotient $21\left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right) \div 7\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$. Then express the quotient in rectangular form.**

Find the modulus and amplitude of the quotient.

$$r = \frac{r_1}{r_2} = \frac{21}{7} = 3 \qquad \theta = \theta_1 - \theta_2 = \frac{7\pi}{6} - \frac{4\pi}{3} = -\frac{\pi}{6}$$

The quotient is $3\left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right]$.

Now find the rectangular form of the quotient.

$$3\left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right] = 3\left[\frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right)i\right] \qquad \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},\ \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$
$$= \frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

The rectangular form of the quotient is $\frac{3\sqrt{3}}{2} - \frac{3}{2}i$.

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Study Guide

Powers and Roots of Complex Numbers

You can use De Moivre's Theorem, $[r(\cos\theta + \boldsymbol{i}\sin\theta)]^n = r^n(\cos n\theta + \boldsymbol{i}\sin n\theta)$, to find the powers and roots of complex numbers in polar form.

Example 1 **Find $\left(-1 + \sqrt{3}\boldsymbol{i}\right)^3$.**

First, write $-1 + \sqrt{3}\boldsymbol{i}$ in polar form. Note that its graph is in Quadrant II of the complex plane.

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} \text{ or } 2 \qquad \theta = \text{Arctan}\,\frac{\sqrt{3}}{-1} + \pi = -\frac{\pi}{3} + \pi \text{ or } \frac{2\pi}{3}$$

The polar form of $-1 + \sqrt{3}\boldsymbol{i}$ is $2\left(\cos\frac{2\pi}{3} + \boldsymbol{i}\sin\frac{2\pi}{3}\right)$.

Now use De Moivre's Theorem to find the third power.

$$\begin{aligned}
\left(-1 + \sqrt{3}\boldsymbol{i}\right)^3 &= \left[2\left(\cos\frac{2\pi}{3} + \boldsymbol{i}\sin\frac{2\pi}{3}\right)\right]^3 \\
&= 2^3\left[\cos 3\left(\frac{2\pi}{3}\right) + \boldsymbol{i}\sin 3\left(\frac{2\pi}{3}\right)\right] && \textit{De Moivre's Theorem} \\
&= 8(\cos 2\pi + \boldsymbol{i}\sin 2\pi) \\
&= 8(1 + 0\boldsymbol{i}) && \textit{Write the result in} \\
&= 8 && \textit{rectangular form.}
\end{aligned}$$

Therefore, $\left(-1 + \sqrt{3}\boldsymbol{i}\right)^3 = 8$.

Example 2 **Find $\sqrt[3]{64\boldsymbol{i}}$.**

$$\begin{aligned}
\sqrt[3]{64\boldsymbol{i}} &= (0 + 64\boldsymbol{i})^{\frac{1}{3}} && a = 0,\ b = 64 \\
&= \left[64\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right)\right]^{\frac{1}{3}} && \textit{Polar form: } r = \sqrt{0^2 + 64^2} \textit{ or } 64; \\
& && \theta = \frac{\pi}{2} \textit{ since } a = 0. \\
&= 64^{\frac{1}{3}}\left[\cos\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right) + \boldsymbol{i}\sin\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right)\right] && \textit{De Moivre's Theorem} \\
&= 4\left(\cos\frac{\pi}{6} + \boldsymbol{i}\sin\frac{\pi}{6}\right) \\
&= 4\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i}\right) \\
&= 2\sqrt{3} + 2\boldsymbol{i}
\end{aligned}$$

Therefore, $2\sqrt{3} + 2\boldsymbol{i}$ is the principal cube root of $64\boldsymbol{i}$.

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Study Guide

Introduction to Analytic Geometry

Example 1 **Find the distance between points at (−2, 2) and (5, −4). Then find the midpoint of the segment that has endpoints at the given coordinates.**

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ *Distance Formula*

$d = \sqrt{[5 - (-2)]^2 + [(-4) - 2]^2}$ *Let* $(x_1, y_1) = (-2, 2)$ *and* $(x_2, y_2) = (5, -4)$.

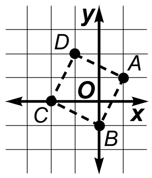
$d = \sqrt{7^2 + (-6)^2}$ or $\sqrt{85}$

midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ *Midpoint Formula*

The midpoint is at $\left(\frac{-2 + 5}{2}, \frac{2 + (-4)}{2}\right)$ or $\left(\frac{3}{2}, -1\right)$.

Example 2 **Determine whether quadrilateral *ABCD* with vertices *A*(1, 1), *B*(0, −1), *C*(−2, 0), and *D*(−1, 2) is a parallelogram.**

First, graph the figure.



To determine if $\overline{DA} \parallel \overline{CB}$, find the slopes of $\overline{DA}$ and $\overline{CB}$.

slope of $\overline{DA}$

$m = \frac{y_2 - y_1}{x_2 - x_1}$ *Slope formula*

$= \frac{1 - 2}{1 - (-1)}$ *D(−1, 2) and A(1, 1)*

$= -\frac{1}{2}$

slope of $\overline{CB}$

$m = \frac{y_2 - y_1}{x_2 - x_1}$ *Slope formula*

$= \frac{-1 - 0}{0 - (-2)}$ *C(−2, 0) and B(0, −1)*

$= -\frac{1}{2}$

Their slopes are equal. Therefore, $\overline{DA} \parallel \overline{CB}$.

To determine if $\overline{DA} \cong \overline{CB}$, use the distance formula to find DA and CB.

$DA = \sqrt{[1 - (-1)]^2 + (1 - 2)^2}$
$= \sqrt{5}$

$CB = \sqrt{[0 - (-2)]^2 + (-1 - 0)^2}$
$= \sqrt{5}$

The measures of $\overline{DA}$ and $\overline{CB}$ are equal. Therefore, $\overline{DA} \cong \overline{CB}$.

Since $\overline{DA} \parallel \overline{CB}$ and $\overline{DA} \cong \overline{CB}$, quadrilateral *ABCD* is a parallelogram.

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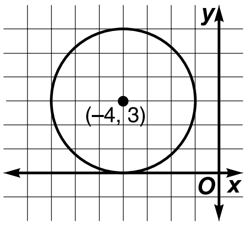
Study Guide

Circles

The standard form of the equation of a **circle** with **radius** r and **center** at (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.

Example 1 **Write the standard form of the equation of the circle that is tangent to the x-axis and has its center at $(-4, 3)$. Then graph the equation.**

Since the circle is tangent to the x-axis, the distance from the center to the x-axis is the radius. The center is 3 units above the x-axis. Therefore, the radius is 3.



$$
\begin{aligned}
(x - h)^2 + (y - k)^2 &= r^2 && \textit{Standard form}\\
[x - (-4)]^2 + (y - 3)^2 &= 3^2 && (h, k) = (-4, 3) \textit{ and } r = 3\\
(x + 4)^2 + (y - 3)^2 &= 9
\end{aligned}
$$

Example 2 **Write the standard form of the equation of the circle that passes through the points at $(1, -1)$, $(5, 3)$, and $(-3, 3)$. Then identify the center and radius of the circle.**

Substitute each ordered pair (x, y) in the general form $x^2 + y^2 + Dx + Ey + F = 0$ to create a system of equations.

$$
\begin{aligned}
&(1)^2 + (-1)^2 + D(1) + E(-1) + F = 0 && (x, y) = (1, -1)\\
&(5)^2 + (3)^2 + D(5) + E(3) + F = 0 && (x, y) = (5, 3)\\
&(-3)^2 + (3)^2 + D(-3) + E(3) + F = 0 && (x, y) = (-3, 3)
\end{aligned}
$$

Simplify the system of equations.

$$
\begin{aligned}
&D - E + F + 2 = 0\\
&5D + 3E + F + 34 = 0\\
&-3D + 3E + F + 18 = 0
\end{aligned}
$$

The solution to the system is $D = -2$, $E = -6$, and $F = -6$.

The general form of the equation of the circle is $x^2 + y^2 - 2x - 6y - 6 = 0$.

$$
\begin{aligned}
x^2 + y^2 - 2x - 6y - 6 &= 0\\
(x^2 - 2x + ?) + (y^2 - 6y + ?) &= 6 && \textit{Group to form perfect square trinomials.}\\
(x^2 - 2x + 1) + (y^2 - 6y + 9) &= 6 + 1 + 9 && \textit{Complete the square.}\\
(x - 1)^2 + (y - 3)^2 &= 16 && \textit{Factor the trinomials.}
\end{aligned}
$$

After completing the square, the standard form of the circle is $(x - 1)^2 + (y - 3)^2 = 16$. Its center is at $(1, 3)$, and its radius is 4.

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Study Guide

Ellipses

The standard form of the equation of an **ellipse** is

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ when the **major axis** is horizontal.

In this case, a^2 is in the denominator of the x term. The standard form is $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$ when the major axis is vertical. In this case, a^2 is in the denominator of the y term. In both cases, $c^2 = a^2 - b^2$.

Example **Find the coordinates of the center, the foci, and the vertices of the ellipse with the equation $4x^2 + 9y^2 + 24x - 36y + 36 = 0$. Then graph the equation.**

First write the equation in standard form.

$$4x^2 + 9y^2 + 24x - 36y + 36 = 0$$

$$4(x^2 + 6x + ?) + 9(y^2 - 4y + ?) = -36 + ? + ? \qquad \textit{GCF of x terms is 4; GCF of y terms is 9.}$$

$$4(x^2 + 6x + 9) + 9(y^2 - 4y + 4) = -36 + 4(9) + 9(4) \qquad \textit{Complete the square.}$$

$$4(x+3)^2 + 9(y-2)^2 = 36 \qquad \textit{Factor.}$$

$$\frac{(x+3)^2}{9} + \frac{(y+2)^2}{4} = 1 \qquad \textit{Divide each side by 36.}$$

Now determine the values of a, b, c, h, and k. In all ellipses, $a^2 > b^2$. Therefore, $a^2 = 9$ and $b^2 = 4$. Since a^2 is the denominator of the x term, the major axis is parallel to the x-axis.

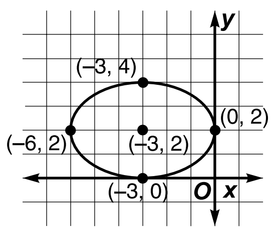
$a = 3 \quad b = 2 \quad c = \sqrt{a^2 - b^2}$ or $\sqrt{5} \quad h = -3 \quad k = 2$

center: $(-3, 2)$ — (h, k)

foci: $(-3 \pm \sqrt{5}, 2)$ — $(h \pm c, k)$

major axis vertices:
$(0, 2)$ and $(-6, 2)$ — $(h \pm a, k)$

minor axis vertices:
$(-3, 4)$ and $(-3, 0)$ — $(h, k \pm b)$



Graph these ordered pairs. Then complete the ellipse.

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Study Guide

Hyperbolas

The standard form of the equation of a **hyperbola** is
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ when the **transverse axis** is horizontal, and
$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ when the transverse axis is vertical. In both cases, $b^2 = c^2 - a^2$.

Example **Find the coordinates of the center, foci, and vertices, and the equations of the asymptotes of the graph of $25x^2 - 9y^2 + 100x - 54y - 206 = 0$. Then graph the equation.**

Write the equation in standard form.

$$25x^2 - 9y^2 + 100x - 54y - 206 = 0$$

$$25(x^2 + 4x + ?) - 9(y^2 + 6y + ?) = 206 + ? + ? \quad \textit{GCF of x terms is 25; GCF of y terms is 9.}$$

$$25(x^2 + 4x + 4) - 9(y^2 + 6y + 9) = 206 + 25(4) + (-9)(9) \quad \textit{Complete the square.}$$

$$25(x + 2)^2 - 9(y + 3)^2 = 225 \quad \textit{Factor.}$$

$$\frac{(x+2)^2}{9} - \frac{(y+3)^2}{25} = 1 \quad \textit{Divide each side by 225.}$$

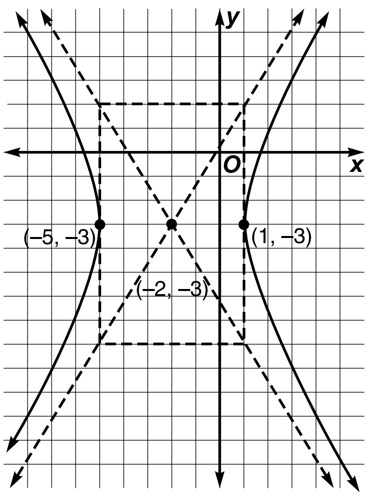
From the equation, $h = -2$, $k = -3$, $a = 3$, $b = 5$, and $c = \sqrt{34}$. The center is at $(-2, -3)$.

Since the x terms are in the first expression, the hyperbola has a horizontal transverse axis.

The vertices are at $(h \pm a, k)$ or $(1, -3)$ and $(-5, -3)$.

The foci are at $(h \pm c, k)$ or $(-2 \pm \sqrt{34}, -3)$.

The equations of the asymptotes are $y - k = \pm\frac{b}{a}(x - h)$ or $y + 3 = \pm\frac{5}{3}(x + 2)$.



Graph the center, the vertices, and the rectangle guide, which is $2a$ units by $2b$ units. Next graph the asymptotes. Then sketch the hyperbola.

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Study Guide

Parabolas

The standard form of the equation of the **parabola** is $(y - k)^2 = 4p(x - h)$ when the parabola opens to the right. When p is negative, the parabola opens to the left. The standard form is $(x - h)^2 = 4p(y - k)$ when the parabola opens upward. When p is negative, the parabola opens downward.

Example 1 **Given the equation $x^2 = 12y + 60$, find the coordinates of the focus and the vertex and the equations of the directrix and the axis of symmetry. Then graph the equation of the parabola.**

First write the equation in the form $(x - h)^2 = 4p(y - k)$.

$x^2 = 12y + 60$

$x^2 = 12(y + 5)$ *Factor.*

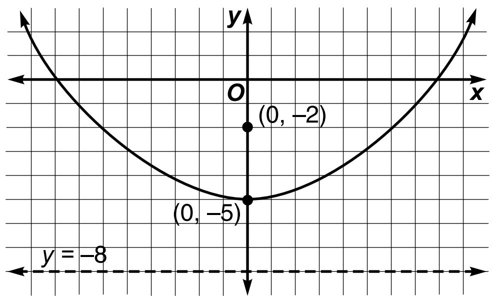
$(x - 0)^2 = 4(3)(y + 5)$ *4p = 12, so p = 3.*

In this form, we can see that $h = 0$, $k = -5$, and $p = 3$.

Vertex: $(0, -5)$ (h, k) Focus: $(0, -2)$ $(h, k + p)$

Directrix: $y = -8$ $y = k - p$ Axis of Symmetry: $x = 0$ $x = h$

The axis of symmetry is the y-axis. Since p is positive, the parabola opens upward. Graph the directrix, the vertex, and the focus. To determine the shape of the parabola, graph several other ordered pairs that satisfy the equation and connect them with a smooth curve.



Example 2 **Write the equation $y^2 + 6y + 8x + 25 = 0$ in standard form. Find the coordinates of the focus and the vertex, and the equations of the directrix and the axis of symmetry. Then graph the parabola.**

$y^2 + 6y + 8x + 25 = 0$

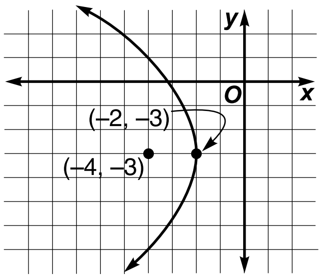
$y^2 + 6y = -8x - 25$ *Isolate the x terms and the y terms.*

$y^2 + 6y + ? = -8x - 25 + ?$

$y^2 + 6y + 9 = -8x - 25 + 9$ *Complete the square.*

$(y + 3)^2 = -8(x + 2)$ *Simplify and factor.*

From the standard form, we can see that $h = -2$ and $k = -3$. Since $4p = -8$, $p = -2$. Since y is squared, the directrix is parallel to the y-axis. The axis of symmetry is the x-axis. Since p is negative, the parabola opens to the left.



Vertex: $(-2, -3)$ (h, k)

Focus: $(-4, -3)$ $(h + p, k)$

Directrix: $x = 0$ $x = h - p$

Axis of Symmetry: $y = -3$ $y = k$

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Study Guide

Rectangular and Parametric Forms of Conic Sections

Use the table to identify a conic section given its equation in general form.

conic	$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
circle	$A = C$
parabola	Either A or C is zero.
ellipse	A and C have the same sign and $A \neq C$.
hyperbola	A and C have opposite signs.

Example 1 **Identify the conic section represented by the equation $5x^2 + 4y^2 - 10x - 8y + 18 = 0$.**

$A = 5$ and $C = 4$. Since A and C have the same signs and are not equal, the conic is an ellipse.

Example 2 **Find the rectangular equation of the curve whose parametric equations are $x = 2t$ and $y = 4t^2 + 4t - 1$. Then identify the conic section represented by the equation.**

First, solve the equation $x = 2t$ for t.

$$x = 2t$$
$$\frac{x}{2} = t$$

Then substitute $\frac{x}{2}$ for t in the equation $y = 4t^2 + 4t - 1$.

$$y = 4t^2 + 4t - 1$$
$$y = 4\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right) - 1 \qquad t = \frac{x}{2}$$
$$y = x^2 + 2x - 1$$

Since $C = 0$, the equation $y = x^2 + 2x - 1$ is the equation of a parabola.

Example 3 **Find the rectangular equation of the curve whose parametric equations are $x = 3 \cos t$ and $y = 5 \sin t$, where $0 \leq t \leq 2\pi$. Then graph the equation using arrows to indicate orientation.**

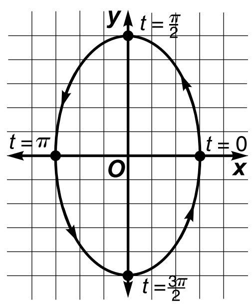
Solve the first equation for $\cos t$ and the second equation for $\sin t$.

$\cos t = \frac{x}{3}$ and $\sin t = \frac{y}{5}$

Use the trigonometric identity $\cos^2 t + \sin^2 t = 1$ to eliminate t.

$$\cos^2 t + \sin^2 t = 1$$
$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1 \qquad \textit{Substitution}$$
$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

This is the equation of an ellipse with the center at (0, 0). As t increases from 0 to 2π, the curve is traced in a counterclockwise motion.



Study Guide

Transformation of Conics

Translations are often written in the form $T_{(h,\,k)}$. To find the equation of a rotated conic, replace x with $x' \cos\theta + y' \sin\theta$ and y with $-x' \sin\theta + y' \cos\theta$.

Example **Identify the graph of each equation. Write an equation of the translated or rotated graph in general form.**

a. $\mathbf{4x^2 + y^2 = 12}$ **for** $\mathbf{T_{(-2,\,3)}}$

The graph of this equation is an ellipse. To write the equation of $4x^2 + y^2 = 12$ for $T_{(-2,\,3)}$, let $h = -2$ and $k = 3$. Then replace x with $x - h$ and y with $y - k$.

$x^2 \Rightarrow (x - (-2))^2$ or $(x + 2)^2$
$y^2 \Rightarrow (y - 3)^2$

Thus, the translated equation is $4(x + 2)^2 + (y - 3)^2 = 12$.

Write the equation in general form.

$$\begin{aligned} 4(x+2)^2 + (y-3)^2 &= 12 & \\ 4(x^2 + 4x + 4) + y^2 - 6y + 9 &= 12 & &\textit{Expand the binomial.} \\ 4x^2 + y^2 + 16x - 6y + 25 &= 12 & &\textit{Simplify.} \\ 4x^2 + y^2 + 16x - 6y + 13 &= 0 & &\textit{Subtract 12 from both sides.} \end{aligned}$$

b. $\mathbf{x^2 - 4y = 0,\ \theta = 45°}$

The graph of this equation is a parabola. Find the expressions to replace x and y.

Replace x with $x' \cos 45° + y' \sin 45°$ or $\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$.

Replace y with $-x' \sin 45° + y' \cos 45°$ or $-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$.

$$\begin{aligned} x^2 - 4y &= 0 & \\ \left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 - 4\left(-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) &= 0 & &\textit{Replace x and y.} \\ \left[\frac{1}{2}(x')^2 + x'y' + \frac{1}{2}(y')^2\right] - 4\left(-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) &= 0 & &\textit{Expand the binomial.} \\ \frac{1}{2}(x')^2 + x'y' + \frac{1}{2}(y')^2 + 2\sqrt{2}x' - 2\sqrt{2}y' &= 0 & &\textit{Simplify.} \end{aligned}$$

The equation of the parabola after the 45° rotation is

$\frac{1}{2}(x')^2 + x'y' + \frac{1}{2}(y')^2 + 2\sqrt{2}x' - 2\sqrt{2}y' = 0$

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Study Guide

Systems of Second-Degree Equations and Inequalities

To find the exact solution to a system of second-degree equations, you must use algebra. Graph systems of inequalities involving second-degree equations to find solutions for the inequality.

Example **a. Solve the system of equations algebraically. Round to the nearest tenth.**

$\boldsymbol{x^2 + 2y^2 = 9}$
$\boldsymbol{3x^2 - y^2 = 1}$

Since both equations contain a single term involving y, you can solve the system as follows.

First, multiply each side of the second equation by 2.

$2(3x^2 - y^2) = 2(1)$
$6x^2 - 2y^2 = 2$

Then, add the equations.

$$\begin{array}{r} 6x^2 - 2y^2 = 2 \\ x^2 + 2y^2 = 9 \\ \hline 7x^2 \qquad = 11 \\ x = \pm\sqrt{\frac{11}{7}} \end{array}$$

Now find y by substituting $\pm\sqrt{\frac{11}{7}}$ for x in one of the original equations.

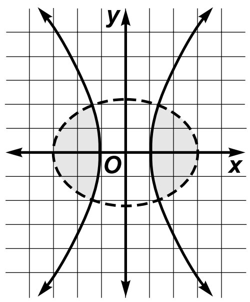
$$x^2 + 2y^2 = 9 \rightarrow \left(\pm\sqrt{\frac{11}{7}}\right)^2 + 2y^2 = 9$$
$$11 + 14y^2 = 63$$
$$y^2 = \frac{26}{7}$$
$$y \approx \pm 1.9$$

The solutions are $(1.3, \pm 1.9)$, and $(-1.3, \pm 1.9)$.

b. Graph the solutions for the system of inequalities.

$\boldsymbol{x^2 + 2y^2 < 9}$
$\boldsymbol{3x^2 - y^2 \geq 1}$

First graph $x^2 + 2y^2 < 9$. The ellipse should be dashed. Test a point either inside or outside the ellipse to see if its coordinates satisfy the inequality. Since $(0, 0)$ satisfies the inequality, shade the interior of the ellipse.



Then graph $3x^2 - y^2 \geq 1$. The hyperbola should be a solid curve. Test a point inside the branches of the hyperbola or outside its branches. Since $(0, 0)$ does not satisfy the inequality, shade the regions inside the branches. The intersection of the two graphs represents the solution of the system.

11-1

Study Guide

Rational Exponents

Example 1 **Simplify each expression.**

a. $\left(\frac{c^5d^3}{c^3d^2}\right)^{\frac{1}{2}}$

$$\left(\frac{c^5d^3}{c^3d^2}\right)^{\frac{1}{2}} = (c^2d)^{\frac{1}{2}} \qquad \frac{a^m}{a^n} = a^{m-n}$$
$$= c^{\frac{2}{2}}d^{\frac{1}{2}} \qquad (a^m)^n = a^{mn}$$
$$= |c|\sqrt{d}$$

b. $\left(\frac{p^2}{q^3}\right)^{-3}$

$$\left(\frac{p^2}{q^3}\right)^{-3} = \frac{p^{-6}}{q^{-9}} \qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$
$$= \frac{q^9}{p^6} \qquad b^{-n} = \frac{1}{b^n}$$

Example 2 **Evaluate each expression.**

a. $64^{\frac{2}{3}}$

$$64^{\frac{2}{3}} = (4^3)^{\frac{2}{3}} \qquad 64 = 4^3$$
$$= 4^2 \text{ or } 16 \qquad (a^m)^n = a^{mn}$$

b. $\frac{27^{\frac{2}{3}}}{27^{\frac{1}{3}}}$

$$\frac{27^{\frac{2}{3}}}{27^{\frac{1}{3}}} = 27^{\frac{2}{3}-\frac{1}{3}} \qquad \frac{a^m}{a^n} = a^{m-n}$$
$$= 27^{\frac{1}{3}} \text{ or } 3$$

c. $\sqrt{35} \cdot \sqrt{10}$

$$\sqrt{35} \cdot \sqrt{10} = 35^{\frac{1}{2}} \cdot 10^{\frac{1}{2}} \qquad \sqrt[n]{b} = b^{\frac{1}{n}}$$
$$= (7 \cdot 5)^{\frac{1}{2}} \cdot (5 \cdot 2)^{\frac{1}{2}}$$
$$= 7^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} \qquad (ab)^m = a^mb^m$$
$$= 7^{\frac{1}{2}} \cdot 5 \cdot 2^{\frac{1}{2}} \qquad a^ma^n = a^{m+n}$$
$$= 5 \cdot \sqrt{7} \cdot \sqrt{2}$$
$$= 5\sqrt{14} \qquad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

Example 3 **Express $\sqrt[3]{8x^6y^{12}}$ using rational exponents.**

$$\sqrt[3]{8x^6y^{12}} = (8x^6y^{12})^{\frac{1}{3}} \qquad b^{\frac{1}{n}} = \sqrt[n]{b}$$
$$= 8^{\frac{1}{3}}x^{\frac{6}{3}}y^{\frac{12}{3}} \qquad (ab)^m = a^mb^m$$
$$= 2x^2y^4$$

Example 4 **Express $16x^{\frac{3}{4}}y^{\frac{1}{2}}$ using radicals.**

$$16x^{\frac{3}{4}}y^{\frac{1}{2}} = 16(x^3y^2)^{\frac{1}{4}} \qquad (ab)^m = a^mb^m$$
$$= 16\sqrt[4]{x^3y^2}$$

11-2

Study Guide

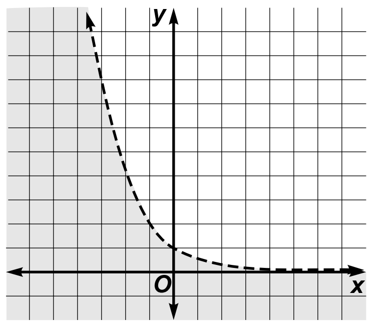
Exponential Functions

Functions of the form $y = b^x$, in which the base b is a positive real number and the exponent is a variable, are known as **exponential functions.** Many real-world situations can be modeled by exponential functions. The equation $N = N_0(1 + r)^t$, where N is the final amount, N_0 is the initial amount, r is the rate of growth or decay, and t is time, is used for modeling exponential growth. The compound interest equation is $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where P is the principal or initial investment, A is the final amount of the investment, r is the annual interest rate, n is the number of times interest is compounded each year, and t is the number of years.

Example 1 **Graph $y < 2^{-x}$.**

First, graph $y = 2^{-x}$. This graph is a function, since there is a unique y-value for each x-value.

x	−3	−2	−1	0	1	2	3	4
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$



Since the points on this curve are not in the solution of the inequality, the graph of $y = 2^{-x}$ is shown as a dashed curve.

Then, use (0, 0) as a test point to determine which area to shade.

$y < 2^{-x}$

$0 < 2^0$

$0 < 1$

Since (0,0) satisfies the inequality, the region that contains (0,0) should be shaded.

Example 2 ***Biology*** **Suppose a researcher estimates that the initial population of a colony of cells is 100. If the cells reproduce at a rate of 25% per week, what is the expected population of the colony in six weeks?**

$N = N_0(1 + r)^t$

$N = 100(1 + 0.25)^6$ *$N_0 = 100$, $r = 0.25$, $t = 6$*

$N \approx 381.4697266$ *Use a calculator.*

There will be about 381 cells in the colony in 6 weeks.

Example 3 ***Finance*** **Determine the amount of money in a money market account that provides an annual rate of 6.3% compounded quarterly if $1700 is invested and left in the account for eight years.**

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

$A = 1700\left(1 + \frac{0.063}{4}\right)^{4 \cdot 8}$ *$P = 1700$, $r = 0.063$, $n = 4$, $t = 8$*

$A \approx 2803.028499$ *Use a calculator.*

After 8 years, the $1700 investment will have a value of $2803.03.

11-3

Study Guide

The Number e

The number $\boldsymbol{e}$ is a special irrational number with an approximate value of 2.718 to three decimal places. The formula for exponential growth or decay is $N = N_0e^{kt}$, where N is the final amount, N_0 is the initial amount, k is a constant, and t is time. The equation $A = Pe^{rt}$, where P is the initial amount, A is the final amount, r is the annual interest rate, and t is time in years, is used for calculating interest that is compounded continuously.

Example 1 ***Demographics*** **The population of Dubuque, Iowa, declined at a rate of 0.4% between 1997 1998. In 1998, the population was 87,806.**

a. Let t be the number of years since 1998 and write a function to model the population.

b. Suppose that the rate of decline remains steady at 0.4%. Find the projected population of Dubuque in 2010.

a. $y = ne^{kt}$

$y = 87{,}806e^{-0.004t}$ *$n = 87{,}806;\ k = -0.004$*

b. In 2010, it will have been 2010 − 1998 or 12 years since the initial population figure. Thus, $t = 12$.

$y = 87{,}806e^{-0.004t}$

$y = 87{,}806e^{-0.004(12)}$ *$t = 12$*

$y \approx 83690.86531$ *Use a calculator.*

Given a population of 87,806 in 1998 and a steady rate of decline of 0.4%, the population of Dubuque, Iowa, will be approximately 83,691 in 2010.

Example 2 ***Finance*** **Compare the balance after 10 years of a \$5000 investment earning 8.5% interest compounded continuously to the same investment compounded quarterly.**

In both cases, $P = 5000$, $r = 0.085$, and $t = 10$. When the interest is compounded quarterly, $n = 4$. Use a calculator to evaluate each expression.

Continuously

$A = Pe^{rt}$

$A = 5000e^{(0.085)(10)}$

$A = 11{,}698.23$

Quarterly

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

$A = 5000\left(1 + \frac{0.085}{4}\right)^{4 \cdot 10}$

$A = 11{,}594.52$

You would earn \$11,698.23 − \$11,594.52 = \$103.71 more by choosing the account that compounds continuously.

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Study Guide

Logarithmic Functions

In the function $x = a^y$, y is called the **logarithm** of x. It is usually written as $y = \log_a x$ and is read "y equals the log, base a, of x." Knowing that if $a^u = a^v$ then $u = v$, you can evaluate a logarithmic expression to determine its logarithm.

Example 1 **Write $\log_7 49 = 2$ in exponential form.**

The base is 7 and the exponent is 2.
$7^2 = 49$

Example 2 **Write $2^5 = 32$ in logarithmic form.**

The base is 2, and the exponent or logarithm is 5.
$\log_2 32 = 5$

Example 3 **Evaluate the expression $\log_5 \frac{1}{25}$.**

Let $x = \log_5 \frac{1}{25}$.

$x = \log_5 \frac{1}{25}$

$5^x = \frac{1}{25}$ — *Definition of logarithm.*

$5^x = (25)^{-1}$ — $a^{-m} = \frac{1}{a^m}$

$5^x = (5^2)^{-1}$ — $5^2 = 25$

$5^x = 5^{-2}$ — $(a^m)^n = a^{mn}$

$x = -2$ — *If* $a^u = a^v$, *then* $u = v$.

Example 4 **Solve each equation.**

a. $\log_6 (4x + 6) = \log_6 (8x - 2)$

$\log_6 (4x + 6) = \log_6 (8x - 2)$

$4x + 6 = 8x - 2$ — *If* $\log_b m = \log_b n$, *then* $m = n$.

$-4x = -8$

$x = 2$

b. $\log_9 x + \log_9 (x - 2) = \log_9 3$

$\log_9 x + \log_9 (x - 2) = \log_9 3$

$\log_9 [x(x - 2)] = \log_9 3$ — $\log_b mn = \log_b m + \log_b n$

$x^2 - 2x = 3$ — *If* $\log_b m = \log_b n$, *then* $m = n$.

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$ — *Factor.*

$x - 3 = 0$ or $x + 1 = 0$ — *Find the zeros.*

$x = 3$ or $x = -1$.

The log of a negative value does not exist, so the answer is $x = 3$.

11-5 Study Guide

Common Logarithms

Logarithms with base 10 are called **common logarithms.** The change of base formula, $\log_a n = \frac{\log_b n}{\log_b a}$, where a, b, and n are positive numbers and neither a nor b is 1, allows you to evaluate logarithms in other bases with a calculator. Logarithms can be used to solve **exponential equations.**

Example 1 **Evaluate each expression.**

a. $\log 8(3)^2$

$\log 8(3)^2 = \log 8 + 2 \log 3$	*$\log ab = \log a + \log b$, $\log b^n = n \log b$*
$\approx 0.9031 + 2(0.4771)$	*Use a calculator.*
$\approx 0.9031 + 0.9542$	
≈ 1.8573	

b. $\log \frac{15^3}{7}$

$\log \frac{15^3}{7} = 3 \log 15 - \log 7$	*$\log \frac{a}{b} = \log a - \log b$, $\log a^m = m \log a$*
$\approx 3(1.1761) - 0.8451$	*Use a calculator.*
$\approx 3.5283 - 0.8451$	
≈ 2.6832	

Example 2 **Find the value of $\log_8 2037$ using the change of base formula.**

$\log_8 2037 = \frac{\log_{10} 2037}{\log_{10} 8}$	*$\log_a n = \frac{\log_b n}{\log_b a}$*
$\approx \frac{3.3090}{0.9031}$	*Use a calculator.*
≈ 3.6641	

Example 3 **Solve $7^{2x} = 93$.**

$7^{2x} = 93$	
$\log 7^{2x} = \log 93$	*Take the logarithm of each side.*
$2x \log 7 = \log 93$	*$\log_b m^p = p \cdot \log_b m$*
$2x = \frac{\log 93}{\log 7}$	*Divide each side by log 7.*
$2x \approx 2.3293$	*Use a calculator.*
$x \approx 1.1646$	

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Study Guide

Natural Logarithms

Logarithms with base e are called **natural logarithms** and are usually written **ln x.** Logarithms with a base other than e can be converted to natural logarithms using the change of base formula. The antilogarithm of a natural logarithm is written **antiln x.** You can use the properties of logarithms and antilogarithms to simplify and solve exponential and logarithmic equations or inequalities with natural logarithms.

Example 1 **Convert $\log_4 381$ to a natural logarithm and evaluate.**

$$\log_a n = \frac{\log_b n}{\log_b a}$$

$$\log_4 381 = \frac{\log_e 381}{\log_e 4} \qquad a = 4,\ b = e,\ n = 381$$

$$= \frac{\ln 381}{\ln 4} \qquad \log_e x = \ln x$$

$$\approx 4.2868 \qquad \textit{Use a calculator.}$$

So, $\log_4 381$ is about 4.2868.

Example 2 **Solve $3.75 = -7.5 \ln x$.**

$$3.75 = -7.5 \ln x$$

$$-0.5 = \ln x \qquad \textit{Divide each side by } -7.5$$

$$\text{antiln}\,(-0.5) = x \qquad \textit{Take the antilogarithm of each side.}$$

$$0.6065 \approx x \qquad \textit{Use a calculator.}$$

The solution is about 0.6065.

Example 3 **Solve each equation or inequality by using natural logarithms.**

a. $4^{3x} = 6^{x+1}$

$$4^{3x} = 6^{x+1}$$

$$\ln 4^{3x} = \ln 6^{x+1} \qquad \textit{Take the natural logarithm of each side.}$$

$$3x \ln 4 = (x + 1) \ln 6 \qquad \ln a^n = n \ln a$$

$$3x(1.3863) = (x + 1)(1.7918) \qquad \textit{Use a calculator.}$$

$$4.1589x = 1.7918x + 1.7918$$

$$2.3671x = 1.7918$$

$$x \approx 0.7570$$

b. $25 > e^{0.2t}$

$$25 > e^{0.2t}$$

$$\ln 25 > \ln e^{0.2t} \qquad \textit{Take the natural logarithm of each side.}$$

$$\ln 25 > 0.2t \ln e \qquad \ln a^n = n \ln a$$

$$3.2189 > 0.2t \qquad \textit{Use a calculator.}$$

$$16.0945 > t$$

Thus, $t < 16.0945$

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Study Guide

Modeling Real-World Data with Exponential and Logarithmic Functions

The **doubling time,** or amount of time t required for a quantity modeled by the exponential equation $N = N_0e^{kt}$ to double, is given by $t = \frac{\ln 2}{k}$.

Example ***Finance*** **Tara's parents invested $5000 in an account that earns 11.5% compounded continuously. They would like to double their investment in 5 years to help finance Tara's college education.**

a. Will the initial investment of $5000 double within 5 years?

Find the doubling time for the investment. For continuously compounded interest, the constant k is the interest rate written as a decimal.

$$t = \frac{\ln 2}{k}$$

$$= \frac{\ln 2}{0.115} \quad \textit{The decimal for 11.5\% is 0.115.}$$

$$\approx 6.03 \text{ years} \quad \textit{Use a calculator.}$$

Five years is not enough time for the initial investment to double.

b. What interest rate is required for an investment with continuously compounded interest to double in 5 years?

$$t = \frac{\ln 2}{k}$$

$$5 = \frac{\ln 2}{k}$$

$$\frac{1}{5} = \frac{k}{\ln 2} \quad \textit{Take the reciprocal of each side.}$$

$$\frac{\ln 2}{5} = k \quad \textit{Multiply each side by ln 2 to solve for k.}$$

$$0.1386 \approx k$$

An interest rate of 13.9% is required for an investment with continuously compounded interest to double in 5 years.

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Study Guide

Arithmetic Sequences and Series

A **sequence** is a function whose domain is the set of natural numbers. The **terms** of a sequence are the range elements of the function. The difference between successive terms of an **arithmetic sequence** is a constant called the **common difference**, denoted as d. An **arithmetic series** is the indicated sum of the terms of an arithmetic sequence.

Example 1 **a. Find the next four terms in the arithmetic sequence −7, −5, −3, ...**
b. Find the 38th term of this sequence.

a. Find the common difference.
$a_2 - a_1 = -5 - (-7)$ or 2

The common difference is 2. Add 2 to the third term to get the fourth term, and so on.
$a_4 = -3 + 2$ or -1 $\quad a_5 = -1 + 2$ or 1
$a_6 = 1 + 2$ or 3 $\quad a_7 = 3 + 2$ or 5

The next four terms are −1, 1, 3, and 5.

b. Use the formula for the nth term of an arithmetic sequence.
$a_n = a_1 + (n - 1)d$
$a_{38} = -7 + (38 - 1)2 \quad n = 38, a_1 = -7, d = 2$
$a_{38} = 67$

Example 2 **Write an arithmetic sequence that has three arithmetic means between 3.2 and 4.4.**

The sequence will have the form 3.2, __?__, __?__, __?__, 4.4.

First, find the common difference.
$a_n = a_1 + (n - 1)d$
$4.4 = 3.2 + (5 - 1)d \quad n = 5, a_5 = 4.4, a_1 = 3.2$
$4.4 = 3.2 + 4d$
$d = 0.3$

Then, determine the arithmetic means.

a_2	a_3	a_4
3.2 + 0.3 = 3.5	3.5 + 0.3 = 3.8	3.8 + 0.3 = 4.1

The sequence is 3.2, 3.5, 3.8, 4.1, 4.4.

Example 3 **Find the sum of the first 50 terms in the series 11 + 14 + 17 + · · · + 158.**

$S_n = \frac{n}{2}(a_1 + a_n)$
$S_{50} = \frac{50}{2}(11 + 158) \quad n = 50, a_1 = 11, a_{50} = 158$
$= 4225$

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Study Guide

Geometric Sequences and Series

A **geometric sequence** is a sequence in which each term after the first, a_1, is the product of the preceding term and the **common ratio**, r. The terms between two nonconsecutive terms of a geometric sequence are called **geometric means.** The indicated sum of the terms of a geometric sequence is a **geometric series.**

Example 1 **Find the 7th term of the geometric sequence 157, −47.1, 14.13, ...**

First, find the common ratio.

$a_2 \div a_1 = -47.1 \div 157$ or -0.3

The common ratio is -0.3.

Then, use the formula for the nth term of a geometric sequence.

$a_n = a_1 r^{n-1}$

$a_7 = 157(-0.3)^6$ *$n = 7$, $a_1 = 157$, $r = -0.3$*

$a_7 = 0.114453$

The 7th term is 0.114453.

Example 2 **Write a sequence that has two geometric means between 6 and 162.**

The sequence will have the form 6, ___?___, ___?___, 162.

First, find the common ratio.

$a_n = a_1 r^{n-1}$

$162 = 6r^3$ *$a_4 = 162$, $a_1 = 6$, $n = 4$*

$27 = r^3$ *Divide each side by 6.*

$3 = r$ *Take the cube root of each side.*

Then, determine the geometric sequence.

$a_2 = 6 \cdot 3$ or 18 $a_3 = 18 \cdot 3$ or 54

The sequence is 6, 18, 54, 162.

Example 3 **Find the sum of the first twelve terms of the geometric series $12 - 12\sqrt{2} + 24 - 24\sqrt{2} + \cdots$.**

First, find the common ratio.

$a_2 \div a_1 = -12\sqrt{2} \div 12$ or $-\sqrt{2}$

The common ratio is $-\sqrt{2}$.

$S_n = \dfrac{a_1 - a_1 r^n}{1 - r}$

$S_{12} = \dfrac{12 - 12(-\sqrt{2})^{12}}{1 - (-\sqrt{2})}$ *$n = 12$, $a_1 = 12$, $r = -\sqrt{2}$*

$S_{12} = 756(1 - \sqrt{2})$ *Simplify.*

The sum of the first twelve terms of the series is $756(1 - \sqrt{2})$.

Study Guide

Infinite Sequences and Series

An **infinite sequence** is one that has infinitely many terms. An **infinite series** is the indicated sum of the terms of an infinite sequence.

Example 1 **Find $\lim\limits_{n\to\infty} \frac{4n^2 - n + 3}{n^2 + 1}$.**

Divide each term in the numerator and the denominator by the highest power of n to produce an equivalent expression. In this case, n^2 is the highest power.

$$\lim_{n\to\infty} \frac{4n^2 - n + 3}{n^2 + 1} = \lim_{n\to\infty} \frac{\frac{4n^2}{n^2} - \frac{n}{n^2} + \frac{3}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{4 - \frac{1}{n} + \frac{3}{n^2}}{1 + \frac{1}{n^2}} \qquad \textit{Simplify.}$$

$$= \frac{\lim\limits_{n\to\infty} 4 - \lim\limits_{n\to\infty} \frac{1}{n} + \lim\limits_{n\to\infty} 3 \cdot \lim\limits_{n\to\infty} \frac{1}{n^2}}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n^2}} \qquad \textit{Apply limit theorems.}$$

$$= \frac{4 - 0 + 3 \cdot 0}{1 + 0} \text{ or } 4 \qquad \lim_{n\to\infty} 4 = 4,\ \lim_{n\to\infty} \frac{1}{n} = 0,\ \lim_{n\to\infty} 3 = 3,\ \lim_{n\to\infty} \frac{1}{n^2} = 0,\ \lim_{n\to\infty} 1 = 1$$

Thus, the limit is 4.

Example 2 **Find the sum of the series $\frac{3}{2} - \frac{3}{8} + \frac{3}{32} - \cdots$.**

In the series $a_1 = \frac{3}{2}$ and $r = -\frac{1}{4}$.

Since $|r| < 1$, $S = \frac{a_1}{1 - r}$.

$$S = \frac{a_1}{1 - r} = \frac{\frac{3}{2}}{1 - \left(-\frac{1}{4}\right)} \qquad a_1 = \frac{3}{2} \textit{ and } r = -\frac{1}{4}$$

$$= \frac{12}{10} \text{ or } 1\frac{1}{5}$$

The sum of the series is $1\frac{1}{5}$.

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Study Guide

Convergent and Divergent Series

If an infinite series has a sum, or limit, the series is **convergent.** If a series is not convergent, it is **divergent.** When a series is neither arithmetic nor geometric and all the terms are positive, you can use the **ratio test** or the **comparison test** to determine whether the series is convergent or divergent.

Ratio Test	Let a_n and a_{n+1} represent two consecutive terms of a series of positive terms. Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. The series is convergent if $r < 1$ and divergent if $r > 1$. If $r = 1$, the test provides no information.

Comparison Test	• A series of positive terms is convergent if, for $n > 1$, each term of the series is equal to or less than the value of the corresponding term of some convergent series of positive terms. • A series of positive terms is divergent if, for $n > 1$, each term of the series is equal to or greater than the value of the corresponding term of some divergent series of positive terms.

Example 1 **Use the ratio test to determine whether the series $\frac{1\cdot 2}{2^1} + \frac{2\cdot 3}{2^2} + \frac{3\cdot 4}{2^3} + \frac{4\cdot 5}{2^4} + \cdots$ is convergent or divergent.**

First, find the nth term. Then use the ratio test.

$$a_n = \frac{n(n+1)}{2^n} \qquad a_{n+1} = \frac{(n+1)(n+2)}{2^{n+1}}$$

$$r = \lim_{n\to\infty} \frac{\frac{(n+1)(n+2)}{2^{n+1}}}{\frac{n(n+1)}{2^n}}$$

$$r = \lim_{n\to\infty} \frac{(n+1)(n+2)}{2^{n+1}} \cdot \frac{2^n}{n(n+1)}$$ *Multiply by the reciprocal of the divisor.*

$$r = \lim_{n\to\infty} \frac{n+2}{2n}$$ $\frac{2^n}{2^{n+1}} = \frac{1}{2}$

$$r = \lim_{n\to\infty} \frac{1 + \frac{2}{n}}{2}$$ *Divide by the highest power of n and apply limit theorems.*

$$r = \frac{1}{2}$$ *Since $r < 1$, the series is convergent.*

Example 2 **Use the comparison test to determine whether the series $\frac{1}{4^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{13^2} + \cdots$ is convergent or divergent.**

The general term of the series is $\frac{1}{(3n+1)^2}$. The general term of the convergent series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ is $\frac{1}{n^2}$. Since $\frac{1}{(3n+1)^2} < \frac{1}{n^2}$ for all $n \geq 1$, the series $\frac{1}{4^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{13^2} + \cdots$ is also convergent.

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Study Guide

Sigma Notation and the *n*th Term

A series may be written using **sigma notation.**

$$\underset{\text{starting value of } n \rightarrow}{} \sum_{n=1}^{k} a_n$$

maximum value of n → k

starting value of n → $n=1$

a_n ← *expression for general term*

n ↑ *index of summation*

Example 1 **Write each expression in expanded form and then find the sum.**

a. $\sum_{n=1}^{5} (n + 2)$

First, write the expression in expanded form.

$$\sum_{n=1}^{5} (n + 2) = (1 + 2) + (2 + 2) + (3 + 2) + (4 + 2) + (5 + 2)$$

Then, find the sum by simplifying the expanded form. $3 + 4 + 5 + 6 + 7 = 25$

b. $\sum_{m=1}^{\infty} 2\left(\frac{1}{4}\right)^m$

$$\sum_{m=1}^{\infty} 2\left(\frac{1}{4}\right)^m = 2\left(\frac{1}{4}\right)^1 + 2\left(\frac{1}{4}\right)^2 + 2\left(\frac{1}{4}\right)^3 + \cdots$$

$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$$

This is an infinite series. Use the formula $S = \frac{a_1}{1 - r}$.

$$S = \frac{\frac{1}{2}}{1 - \frac{1}{4}} \quad a_1 = \frac{1}{2}, r = \frac{1}{4}$$

$$S = \frac{2}{3}$$

Example 2 **Express the series $26 + 37 + 50 + 65 + \cdots + 170$ using sigma notation.**

Notice that each term is one more than a perfect square. Thus, the nth term of the series is $n^2 + 1$. Since $5^2 + 1 = 26$ and $13^2 + 1 = 170$, the index of summation goes from $n = 5$ to $n = 13$.

Therefore, $26 + 37 + 50 + 65 + \cdots + 170 = \sum_{n=5}^{13} (n^2 + 1)$.

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Study Guide

The Binomial Theorem

Two ways to expand a binomial are to use either **Pascal's triangle** or the **Binomial Theorem.** The Binomial Theorem states that if n is a positive integer, then the following is true.

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{1 \cdot 2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}y^3 + \cdots + y^n$$

To find individual terms of an expansion, use this form of the Binomial Theorem:

$$(x + y)^n = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!}x^{n-r}y^r.$$

Example 1 **Use Pascal's triangle to expand $(x + 2y)^5$.**

First, write the series without the coefficients. The expression should have 5 + 1, or 6, terms, with the first term being x^5 and the last term being y^5. The exponents of x should decrease from 5 to 0 while the exponents of y should increase from 0 to 5. The sum of the exponents of each term should be 5.

$x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ $\qquad$ *$x^0 = 1$ and $y^0 = 1$*

Replace each y with $2y$.

$x^5 + x^4(2y) + x^3(2y)^2 + x^2(2y)^3 + x(2y)^4 + (2y)^5$

Then, use the numbers in the sixth row of Pascal's triangle as the coefficients of the terms, and simplify each term.

$$\begin{aligned}(x + 2y)^5 &= \overset{\overset{1}{\downarrow}}{x^5} + \overset{\overset{5}{\downarrow}}{5}x^4(2y) + \overset{\overset{10}{\downarrow}}{10}x^3(2y)^2 + \overset{\overset{10}{\downarrow}}{10}x^2(2y)^3 + \overset{\overset{5}{\downarrow}}{5}x(2y)^4 + \overset{\overset{1}{\downarrow}}{(2y)^5}\\ &= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5\end{aligned}$$

Example 2 **Find the fourth term of $(5a + 2b)^6$.**

$$(5a + 2b)^6 = \sum_{r=0}^{6} \frac{6!}{r!(6-r)!}(5a)^{6-r}(2b)^r$$

To find the fourth term, evaluate the general term for $r = 3$. *Since r increases from 0 to n, r is one less than the number of the term.*

$$\begin{aligned}\frac{6!}{r!(6-r)!}(5a)^{6-r}(2b)^r &= \frac{6!}{3!(6-3)!}(5a)^{6-3}(2b)^3\\ &= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!3!}(5a)^3(2b)^3\\ &= 20{,}000a^3b^3\end{aligned}$$

The fourth term of $(5a + 2b)^6$ is $20{,}000a^3b^3$.

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Study Guide

Special Sequences and Series

The value of e^x can be approximated by using the **exponential series.** The **trigonometric series** can be used to approximate values of the trigonometric functions. **Euler's formula** can be used to write the exponential form of a complex number and to find a complex number that is the natural logarithm of a negative number.

Example 1 **Use the first five terms of the trigonometric series to approximate the value of sin $\frac{\pi}{6}$ to four decimal places.**

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

Let $x = \frac{\pi}{6}$, or about 0.5236.

$$\sin \frac{\pi}{6} \approx 0.5236 - \frac{(0.5236)^3}{3!} + \frac{(0.5236)^5}{5!} - \frac{(0.5236)^7}{7!} + \frac{(0.5236)^9}{9!}$$

$$\sin \frac{\pi}{6} \approx 0.5236 - 0.02392 + 0.00033 - 0.000002 + 0.000000008$$

$\sin \frac{\pi}{6} \approx 0.5000$ *Compare this result to the actual value, 0.5.*

Example 2 **Write $4 - 4i$ in exponential form.**

Write the polar form of $4 - 4\boldsymbol{i}$.
Recall that $a + b\boldsymbol{i} = r(\cos\theta + \boldsymbol{i}\sin\theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = \text{Arctan}\,\frac{b}{a}$ when $a > 0$.

$r = \sqrt{4^2 + (-4)^2}$ or $4\sqrt{2}$, and *$a = 4$ and $b = -4$*

$\theta = \text{Arctan}\,\frac{-4}{4}$ or $-\frac{\pi}{4}$

$$4 - 4\boldsymbol{i} = 4\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + \boldsymbol{i}\sin\left(-\frac{\pi}{4}\right)\right]$$
$$= 4\sqrt{2}e^{-\boldsymbol{i}\frac{\pi}{4}}$$

Thus, the exponential form of $4 - 4\boldsymbol{i}$ is $4\sqrt{2}e^{-\boldsymbol{i}\frac{\pi}{4}}$.

Example 3 **Evaluate ln(−12.4).**

$\ln(-12.4) = \ln(-1) + \ln(12.4)$
$\approx \boldsymbol{i}\pi + 2.5177$ *Use a calculator to compute ln(12.4).*

Thus, $\ln(-12.4) \approx \boldsymbol{i}\pi + 2.5177$. *The logarithm is a complex number.*

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Study Guide

Sequences and Iteration

Each output of composing a function with itself is called an *iterate*. To iterate a function $f(x)$, find the function value $f(x_0)$ of the initial value x_0. The second iterate is the value of the function performed on the output, and so on.

The function $f(z) = z^2 + c$, where c and z are complex numbers, is central to the study of **fractal geometry.** This type of geometry can be used to describe things such as coastlines, clouds, and mountain ranges.

Example 1 **Find the first four iterates of the function $f(x) = 4x + 1$ if the initial value is -1.**

$x_0 = -1$
$x_1 = 4(-1) + 1$ or -3
$x_2 = 4(-3) + 1$ or -11
$x_3 = 4(-11) + 1$ or -43
$x_4 = 4(-43) + 1$ or -171

The first four iterates are -3, -11, -43, and -171.

Example 2 **Find the first three iterates of the function $f(z) = 3z - \boldsymbol{i}$ if the initial value is $1 + 2\boldsymbol{i}$.**

$z_0 = 1 + 2\boldsymbol{i}$
$z_1 = 3(1 + 2\boldsymbol{i}) - \boldsymbol{i}$ or $3 + 5\boldsymbol{i}$
$z_2 = 3(3 + 5\boldsymbol{i}) - \boldsymbol{i}$ or $9 + 14\boldsymbol{i}$
$z_3 = 3(9 + 14\boldsymbol{i}) - \boldsymbol{i}$ or $27 + 41\boldsymbol{i}$

The first three iterates are $3 + 5\boldsymbol{i}$, $9 + 14\boldsymbol{i}$, and $27 + 41\boldsymbol{i}$.

Example 3 **Find the first three iterates of the function $f(z) = z^2 + c$, where $c = 2 - \boldsymbol{i}$ and $z_0 = 1 + \boldsymbol{i}$.**

$$\begin{aligned} z_1 &= (1 + \boldsymbol{i})^2 + 2 - \boldsymbol{i}. \\ &= 1 + \boldsymbol{i} + \boldsymbol{i} + \boldsymbol{i}^2 + 2 - \boldsymbol{i} \\ &= 1 + \boldsymbol{i} + \boldsymbol{i} + (-1) + 2 - \boldsymbol{i} \qquad \boldsymbol{i}^2 = -1 \\ &= 2 + \boldsymbol{i} \end{aligned}$$

$$\begin{aligned} z_2 &= (2 + \boldsymbol{i})^2 + 2 - \boldsymbol{i} \\ &= 4 + 2\boldsymbol{i} + 2\boldsymbol{i} + \boldsymbol{i}^2 + 2 - \boldsymbol{i} \\ &= 4 + 2\boldsymbol{i} + 2\boldsymbol{i} + (-1) + 2 - \boldsymbol{i} \\ &= 5 + 3\boldsymbol{i} \end{aligned}$$

$$\begin{aligned} z_3 &= (5 + 3\boldsymbol{i})^2 + 2 - \boldsymbol{i} \\ &= 25 + 15\boldsymbol{i} + 15\boldsymbol{i} + 9\boldsymbol{i}^2 + 2 - \boldsymbol{i} \\ &= 25 + 15\boldsymbol{i} + 15\boldsymbol{i} + 9(-1) + 2 - \boldsymbol{i} \\ &= 18 + 29\boldsymbol{i} \end{aligned}$$

The first three iterates are $2 + \boldsymbol{i}$, $5 + 3\boldsymbol{i}$, and $18 + 29\boldsymbol{i}$.

Study Guide

Mathematical Induction

A method of proof called **mathematical induction** can be used to prove certain conjectures and formulas. The following example demonstrates the steps used in proving a summation formula by mathematical induction.

Example **Prove that the sum of the first n positive even integers is $n(n + 1)$.**

Here S_n is defined as $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.

1. First, verify that S_n is valid for the first possible case, $n = 1$. Since the first positive even integer is 2 and $1(1 + 1) = 2$, the formula is valid for $n = 1$.

2. Then, assume that S_n is valid for $n = k$.

$S_k \Rightarrow 2 + 4 + 6 + \cdots + 2k = k(k + 1)$. *Replace n with k.*

Next, prove that S_n is also valid for $n = k + 1$.

$S_{k+1} \Rightarrow 2 + 4 + 6 + \cdots + 2k + 2(k + 1)$

$= k(k + 1) + 2(k + 1)$ *Add 2(k + 1) to both sides.*

We can simplify the right side by adding $k(k + 1) + 2(k + 1)$.

$S_{k+1} \Rightarrow 2 + 4 + 6 + \cdots + 2k + 2(k + 1)$

$= (k + 1)(k + 2)$ *(k + 1) is a common factor.*

If $k + 1$ is substituted into the original formula $(n(n + 1))$, the same result is obtained.

$(k + 1)[(k + 1) + 1]$ or $(k + 1)(k + 2)$

Thus, if the formula is valid for $n = k$, it is also valid for $n = k + 1$. Since S_n is valid for $n = 1$, it is also valid for $n = 2$, $n = 3$, and so on. That is, the formula for the sum of the first n positive even integers holds.

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Study Guide

Permutations and Combinations

Use the **Basic Counting Principle** to determine different possibilities for the arrangement of objects. The arrangement of objects in a certain order is called a **permutation**. A **combination** is an arrangement in which order is *not* a consideration.

Example 1 **Eight students on a student council are assigned 8 seats around a U-shaped table.**

a. How many different ways can the students be assigned seats at the table?

Since order is important, this situation is a permutation. The eight students are taken all at once, so the situation can be represented as $P(8, 8)$.

$P(8, 8) = 8!$ $\qquad P(n, n) = n!$

$= 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or 40,320

There are 40,320 ways the students can be seated.

b. How many ways can a president and a vice-president be elected from the eight students?

This is a permutation of 8 students being chosen 2 at a time.

$P(8, 2) = \frac{8!}{(8-2)!}$ $\qquad P(n, r) = \frac{n!}{(n-r)!}$

$= \frac{8 \cdot 7 \cdot 6!}{6!}$ or 56

There are 56 ways a president and vice-president can be chosen.

Example 2 **The Outdoor Environmental Club consists of 20 members, of which 9 are male and 11 are female. Seven members will be selected to form an event-planning committee. How many committees of 4 females and 3 males can be formed?**

Order is not important. There are three questions to consider.

How many ways can 3 males be chosen from 9?
How many ways can 4 females be chosen from 11?
How many ways can 3 males and 4 females be chosen together?

The answer is the product of the combinations $C(9, 3)$ and $C(11, 4)$.

$C(9, 3) \cdot C(11, 4) = \frac{9!}{(9-3)!3!} \cdot \frac{11!}{(11-4)!4!}$ $\qquad C(n, r) = \frac{n!}{(n-r)!r!}$

$= \frac{9!}{6!3!} \cdot \frac{11!}{7!4!}$

$= 84 \cdot 330$ or 27,720

There are 27,720 possible committees.

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Study Guide

Permutations with Repetitions and Circular Permutations

For permutations involving repetitions, the number of permutations of n objects of which p are alike and q are alike is $\frac{n!}{p!q!}$. When n objects are arranged in a circle, there are $\frac{n!}{n}$, or $(n-1)!$, permutations of the objects around the circle. If n objects are arranged relative to a fixed point, then there are $n!$ permutations.

Example 1 **How many 10-letter patterns can be formed from the letters of the word *basketball*?**

The ten letters can be arranged in $P(10, 10)$, or 10!, ways. However, some of these 3,628,800 ways have the same appearance because some of the letters appear more than once.

$\frac{10!}{2!2!2!}$ *There are 2 a's, 2 b's, and 2 l's in basketball.*

$$\frac{10!}{2!2!2!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1}$$

$$= 453{,}600$$

There are 453,600 ten-letter patterns that can be formed from the letters of the word *basketball*.

Example 2 **Six people are seated at a round table to play a game of cards.**

a. Is the seating arrangement around the table a linear or circular permutation? Explain.

b. How many possible seating arrangements are there?

a. The arrangement of people is a circular permutation since the people form a circle around the table.

b. There are 6 people, so the number of arrangements can be described by $(6-1)!$.

$$(6-1)! = 5!$$
$$= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \text{ or } 120$$

There are 120 possible seating arrangements.

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Study Guide

Probability and Odds

The **probability** of an event is the ratio of the number of ways an event can happen to the total number of ways an event can and cannot happen.

Example **A bag contains 3 black, 5 green, and 4 yellow marbles.**

a. What is the probability that a marble selected at random will be green?

The probability of selecting a green marble is written P(green). There are 5 ways to select a green marble from the bag and $3 + 4$, or 7, ways not to select a green marble. So, success $(s) = 5$ and failure $(f) = 7$. Use the probability formula.

$P(\text{green}) = \frac{5}{5+7}$ or $\frac{5}{12}$ $\qquad P(s) = \frac{s}{s+f}$

The probability of selecting a green marble is $\frac{5}{12}$.

b. What is the probability that a marble selected at random will *not* be yellow?

There are 8 ways not to select a yellow marble and 4 ways to select a yellow marble.

$P(\text{not yellow}) = \frac{8}{4+8}$ or $\frac{2}{3}$ $\qquad P(f) = \frac{f}{s+f}$

The probability of not selecting a yellow marble is $\frac{2}{3}$.

c. What is the probability that 2 marbles selected at random will both be black?

Use counting methods to determine the probability. There are $C(3, 2)$ ways to select 2 black marbles out of 3, and $C(12, 2)$ ways to select 2 marbles out of 12.

$$P(\text{2 black marbles}) = \frac{C(3, 2)}{C(12, 2)}$$

$$= \frac{\frac{3!}{1!2!}}{\frac{12!}{10!2!}} \text{ or } \frac{1}{22}$$

The probability of selecting 2 black marbles is $\frac{1}{22}$.

13-4

Study Guide

Probabilities of Compound Events

Example 1 **Using a standard deck of playing cards, find the probability of drawing a king, replacing it, then drawing a second king.**

Since the first card is returned to the deck, the outcome of the second draw is not affected by the first. The events are independent. The probability is the product of each individual probability.

$P(A \text{ and } B) = P(A) \cdot P(B)$ *Let A represent the first draw and B the second draw.*

$P(A) = P(B) = \frac{4}{52} = \frac{1}{13}$ $\frac{\textit{4 kings}}{\textit{52 cards in a standard deck}}$

$P(A \text{ and } B) = \frac{1}{13} \cdot \frac{1}{13} \cdot = \frac{1}{169}$

The probability of selecting a king, replacing it, and then selecting another king is $\frac{1}{169}$.

Example 2 **What is the probability of selecting a yellow or a blue marble from a box of 5 green, 3 yellow, and 2 blue marbles?**

A yellow marble and a blue marble cannot be selected at the same time. Thus, the events are mutually exclusive. Find the sum of the individual probabilities.

$P(\text{yellow or blue}) = P(\text{yellow}) + P(\text{blue})$

$= \frac{3}{10} + \frac{2}{10}$ $P(\textit{yellow}) = \frac{3}{10}$; $P(\textit{blue}) = \frac{2}{10}$

$= \frac{5}{10} \text{ or } \frac{1}{2}$

Example 3 **What is the probability that a card drawn from a standard deck is either a face card or black?**

The card drawn could be both a face card and black, so the events are mutually inclusive.

$P(\text{face card}) = \frac{12}{52}$

$P(\text{black}) = \frac{26}{52}$

$P(\text{face card and black}) = \frac{6}{52}$

$P(\text{face card or black}) = \frac{12}{52} + \frac{26}{52} - \frac{6}{52} = \frac{32}{52} \text{ or } \frac{8}{13}$

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Study Guide

Conditional Probabilities

The **conditional probability** of event A, given event B, is defined as $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$ where $P(B) \neq 0$. In some situations, event A is a subset of event B. In these situations, $P(A|B) = \frac{P(A)}{P(B)}$.

Example **A box contains 3 red pencils and 4 yellow pencils. Three pencils are selected at random. What is the probability that exactly two red pencils are selected if the second pencil is red?**

Sample spaces and reduced sample spaces can be used to help determine the outcomes that satisfy a given condition.

The sample space is $S = \{RRR, RRY, RYR, RYY, YRR, YRY, YYR, YYY\}$ and includes all of the possible outcomes of selecting 3 pencils out of a box of 3 red and 4 yellow pencils.

Event B represents the condition that the second pencil is red.
$B = \{RRR, RRY, YRR, YRY\}$

$P(B) = \frac{4}{8}$ or $\frac{1}{2}$

Event A represents the condition that exactly two of the pencils are red.

$A = \{RRY, RYR, YRR\}$

$P(A) = \frac{3}{8}$

$(A$ and $B)$ is the intersection of A and B.
$(A$ and $B) = \{RRY, YRR\}$.

So, $P(A \text{ and } B) = \frac{2}{8}$ or $\frac{1}{4}$.

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$= \frac{\frac{1}{4}}{\frac{1}{2}} \text{ or } \frac{1}{2}$$

The probability that exactly two pencils are red given that the second pencil is red is $\frac{1}{2}$.

Study Guide

The Binomial Theorem and Probability

Problems that meet the conditions of a **binomial experiment** can be solved using the binomial expansion. Use the Binomial Theorem to find the probability when the number of trials makes working with the binomial expansion unrealistic.

Example 1 **The probability that Misha will win a word game is $\frac{3}{4}$. If Misha plays the game 5 times, what is the probability that he will win exactly 3 games?**

There are 5 games and each game has only two possible outcomes, win W or lose L. These events are independent and the probability is $\frac{3}{4}$ for each game. So this is a binomial experiment.

When $(W + L)^5$ is expanded, the term W^3L^2 represents 3 wins and 2 losses. The coefficient of W^3L^2 is $C(5, 3)$, or 10.

$$
\begin{aligned}
P(\text{exactly 3 wins}) &= 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 \qquad W = \frac{3}{4}, L = \frac{1}{4} \\
&= 10\left(\frac{27}{64}\right)\left(\frac{1}{16}\right) \\
&= \frac{270}{1024} \\
&= \frac{135}{512} \text{ or about } 26.4\%
\end{aligned}
$$

Example 2 **The probability that a computer salesperson will make a sale when approaching a customer is $\frac{1}{2}$. If the salesperson approaches 12 customers, what is the probability that 8 sales will be made?**

Let S be the probability of a sale.
Let N be the probability of not making a sale.

$$(S + N)^{12} = \sum_{r=0}^{12} \frac{12!}{r!(12 - r)!} P_S^{12-r} P_N^{\,r}$$

Making 8 sales means that 4 sales will not be made. So the probability can be found using the term where $r = 4$.

$$
\begin{aligned}
\frac{12!}{4!(12 - 4)!}S^8N^4 &= 495S^8N^4 \\
&= 495\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^4 \qquad S = \frac{1}{2}, N = \frac{1}{2} \\
&= \frac{495}{4096} \text{ or } 0.120849609
\end{aligned}
$$

The probability of making exactly 8 sales is about 12.1%.

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Study Guide

The Frequency Distribution

A **frequency distribution** is a convenient system for organizing large amounts of data. A number of classes are determined, and all values in a class are tallied and grouped together. The most common way of displaying frequency distributions is by using a type of bar graph called a **histogram**.

Example **The number of passengers who boarded planes at 36 airports in the United States in one year are shown below.**

30,526	30,372	26,623	22,722	16,287	15,246	14,807	14,117	14,054
13,547	12,916	12,616	11,906	11,622	11,489	10,828	10,653	10,008
9703	9594	9463	9348	9125	8572	7300	6772	6549
6126	5907	5712	5287	4848	4832	4820	4750	4684

Source: U.S. Department of Transportation

a. Find the range of the data.
The range of the data is 30,526 − 4684 or 25,842.

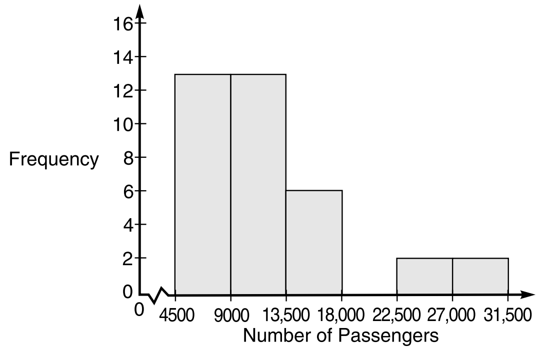
b. Determine an appropriate class interval.
An appropriate class interval is 4500 passengers, beginning with 4500 and ending with 31,500. There will be six classes.

c. Name the class limits and the class marks.
The class limits are the upper and lower values in each interval, or 4500, 9000, 13,500, 18,000, 22,500, 27,000, and 31,500. The class marks are the averages of the class limits of each interval, or 6750, 11,250, 15,750, 20,250, 24,750, and 29,250.

d. Construct a frequency distribution of the data.
Use tallies to determine the number of passengers in each interval.

Number of Passengers	Tallies	Frequency
4500-9000	卌 卌 |||	13
9000-13,500	卌 卌 |||	13
13,500-18,000	卌		6
18,000-22,500		0
22,500-27,000	||	2
27,000-31,500	||	2

e. Draw a histogram of the data.
Label the horizontal axis with the class limits. The vertical axis should be labeled from 0 to a value that will allow for the greatest frequency. Draw the bars side by side so that the height of each bar corresponds to its interval's frequency.



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Study Guide

Measures of Central Tendency

The **mean** is found by adding the values in a set of data and dividing the sum by the number of values in that set. In other words, if a set of data has n values given by X_i such that i is an integer and $1 \leq i \leq n$, then the arithmetic mean $\overline{X}$ can be found as follows.

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

The **median** of a set of data is the middle value. If there are two middle values, the median is the mean of the two middle values. The **mode** of a set of data is the most frequent value. Some sets have multiple modes, and others have no mode.

Example 1 **Find the mean of the set {13, 18, 21, 14, 16, 19, 25, 17}.**

$$\overline{X} = \frac{\text{sum of the values in the set of data}}{\text{number of values in the set}}$$

$$\overline{X} = \frac{13 + 18 + 21 + 14 + 16 + 19 + 25 + 17}{8}$$

$$\overline{X} = \frac{143}{8} \text{ or } 17.875$$

The mean of the set of data is 17.875.

Example 2 **The table at the right shows the number of households without a telephone in 1990.**

State	Number of Households
Alaska	131,600
California	313,100
Florida	270,200
Georgia	195,700
Iowa	36,500
Kentucky	140,900
Minnesota	40,400
Nevada	25,100
New Mexico	67,500
Oklahoma	106,400
West Virginia	70,800

Source: U.S. Census Bureau

a. Find the mean of the data.

Since there are 11 states listed in the table, $n = 11$.

$$\frac{1}{11}\sum_{i=1}^{11} X_i = \frac{1}{11}(131{,}600 + 313{,}100 + 270{,}200 + 195{,}700 + 36{,}500 + 140{,}900 + 40{,}400 + 25{,}100 + 67{,}500 + 106{,}400 + 70{,}800)$$

The mean is about 127,109.

b. Find the median of the data.

To find the median, order the data. Since all the numbers are multiples of 100, you can order the set by hundreds.

251 365 404 675 708 1064 1316 1409 1957 2702 3131

Since there are an odd number of values, the median is the middle value, or 106,400.

c. Find the mode of the data.

Since all elements in the set of data have the same frequency, there is no mode.

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Study Guide

Measures of Variability

If a set of data has been arranged in order and the median is found, the set of data is divided into two groups. If the median of each group is found, the data is divided into four groups. Each of these groups is called a **quartile,** and the quartile points Q_1, Q_2, and Q_3 denote the breaks for each quartile. The **interquartile range** is the difference between the first quartile point and the third quartile point.

Month	Temperature (°F)
Jan.	58.0
Feb.	57.9
March	61.6
April	62.5
May	68.7
June	67.4
July	69.3
Aug.	72.9
Sept.	75.5
Oct.	68.7
Nov.	64.0
Dec.	57.4

Source: National Climatic Data Center

Example 1 **The table shows the average monthly temperatures for San Diego in 1997.**

a. Find the interquartile range of the temperatures and state what it represents.

First, order the data from least to greatest and identify Q_1, Q_2, and Q_3.

57.4 57.9 58.0 61.6 62.5 64.0
67.4 68.7 68.7 69.3 72.9 75.5

For this set of data, the quartile points Q_1, Q_2, and Q_3 are not members of the set. Instead, Q_2 is the mean of the middle values of the set. Thus, $Q_1 = 59.8$, $Q_2 = 65.7$, and $Q_3 = 69.0$. The interquartile range is $69.0 - 59.8$, or 9.2. This means that half the average monthly temperatures are within 9.2°F of each other.

b. Find the semi-interquartile range of the temperatures.

The semi-interquartile range is $\frac{9.2}{2}$, or 4.6.

Example 2 **Find the mean deviation of the temperatures in Example 1.**

There are 12 temperatures listed, and the mean is $\frac{1}{12}\sum_{i=1}^{12} X_i$, or 65.325.

$$MD = \frac{1}{12}\sum_{i=1}^{12} |X_i - 65.325| \qquad MD = \frac{1}{n}\sum_{i=1}^{n} |X_i - \overline{X}|$$

$$MD = \frac{1}{12}(|75.5 - 65.325| + |72.9 - 65.325| + \cdots + |57.4 - 65.325|)$$

$$MD = \frac{1}{12}(|10.175| + |7.575| + \cdots + |-7.925|) \text{ or about } 5.092$$

The mean deviation of the temperatures is about 5.092. This means that the temperatures are an average of about 5.092°F above or below the mean temperature of 65.325°F.

14-4

Study Guide

The Normal Distribution

A **normal distribution** is a frequency distribution that often occurs when there is a large number of values in a set of data. The graph of a normal distribution is a symmetric, bell-shaped curve known as a **normal curve.** The tables below give the fractional parts of a normally distributed set of data for selected areas about the mean. The letter t represents the number of standard deviations from the mean, that is, $\overline{X} \pm t\sigma$. P represents the fractional part that lies in the interval $\overline{X} \pm t\sigma$.

t	P
0.0	0.000
0.1	0.080
0.2	0.159
0.3	0.236
0.4	0.311
0.5	0.383

t	P
0.6	0.451
0.7	0.516
0.8	0.576
0.9	0.632
1.0	0.683
1.1	0.729

t	P
1.2	0.770
1.3	0.807
1.4	0.838
1.5	0.866
1.6	0.891
1.65	0.900

t	P
1.7	0.911
1.8	0.929
1.9	0.943
1.96	0.950
2.0	0.955
2.1	0.964

t	P
2.2	0.972
2.3	0.979
2.4	0.984
2.5	0.988
2.58	0.990
2.6	0.991

t	P
2.7	0.993
2.8	0.995
2.9	0.996
3.0	0.997
3.5	0.9995
4.0	0.9999

Example 1 **Air passengers traveling through Atlanta have an average layover of 82 minutes with a standard deviation of 7.5 minutes. Sketch a normal curve that represents the frequency of layover times.**

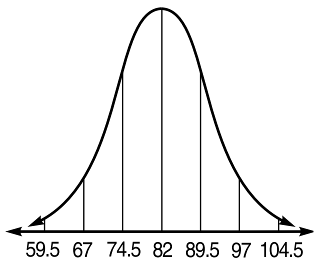
First, find the values defined by the standard deviation in a normal distribution.

$\overline{X} - 1\sigma = 82 - 1(7.5)$ or 74.5 $\quad$ $\overline{X} + 1\sigma = 82 + 1(7.5)$ or 89.5

$\overline{X} - 2\sigma = 82 - 2(7.5)$ or 67 $\quad$ $\overline{X} + 2\sigma = 82 + 2(7.5)$ or 97

$\overline{X} - 3\sigma = 82 - 3(7.5)$ or 59.5 $\quad$ $\overline{X} + 3\sigma = 82 + 3(7.5)$ or 104.5

Then, sketch the general shape of a normal curve. Replace the horizontal scale with the values you have calculated.



Example 2 **Find the upper and lower limits of an interval about the mean within which 15% of the values of a set of normally distributed data can be found if $\overline{X} = 725$ and $\sigma = 4$.**

Use the tables above to find the value of t that most closely approximates $P = 0.15$. For $t = 0.2$, $P = 0.159$. Choose $t = 0.2$.

Now find the limits. $\overline{X} \pm t\sigma = 725 \pm 0.2(4)$ $\quad$ *$\overline{X} = 725$, $t = 0.2$, $\sigma = 4$*

$= 724.2$ and 725.8

The interval in which 15% of the data lies is 724.2–725.8.

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Study Guide

Sample Sets of Data

In statistics, the word **population** refers to an entire set of items or individuals in a group. Rarely will 100% of a population be accessible as a source of data. Therefore, researchers usually select a **random sample** of the population to represent the entire population. Because discrepancies are common in random samples, researchers often take many samples and assume that the sample mean is near its true population mean. The standard deviation of the distribution of the sample means is known as the **standard error of the mean.**

Standard Error of the Mean	If a sample of data has N values and σ is the standard deviation, the standard error of the mean $\sigma_{\bar{x}}$ is $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$.

Example 1 **A sample of data has 4500 values and a standard deviation of 12. What is the standard error of the mean?**

For the sample, $N = 4500$. Find $\sigma_{\bar{x}}$.

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}} = \frac{12}{\sqrt{4500}}$, or about 0.179

The standard error of the mean for the set of data is approximately 0.179.

Example 2 **The daily calorie consumption of people in the United States is normally distributed. A team of nutritionists takes a sample of 250 people and records their daily calorie consumption. From this sample, the average daily calorie consumption is 2150 with a standard deviation of 60 calories per day. Determine the interval about the sample mean that has a 1% level of confidence.**

A 1% level of confidence means that there is less than a 1% chance that the true mean differs from the sample mean by a certain amount. A 1% level of condifence is given when $P = 99\%$. When $P = 0.99$, $t = 2.58$.

Find $\sigma_{\bar{x}}$. $\qquad \sigma_{\bar{x}} = \frac{60}{\sqrt{250}}$ or about 3.795

Find the range. $\qquad \bar{X} = 2150 \pm 2.58(3.795)$

$\bar{X} = 2140.2089$ and 2159.7911

Thus, the probability is 99% that the true mean is within the interval of 2140.2089 calories per day to 2159.7911 calories per day.

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Study Guide

Limits

If the function $f(x)$ is continuous at a, then $\lim_{x \to a} f(x) = f(a)$. Continuous functions include polynomials as well as the functions $\sin x$, $\cos x$, and a^x. Also, $\log_a x$ is continuous if $x > 0$. To evaluate the **limit** when a function is not continuous at the x-value in question, apply algebraic methods to decompose the function into a simpler one. If the function cannot be simplified, compute values of the function using x-values that get closer and closer to a from either side.

Example 1 **Evaluate each limit.**

a. $\lim_{x \to 1} (x^4 - 2x^2 + x + 3)$

Since $f(x) = x^4 - 2x^2 + x + 3$ is a polynomial function, it is continuous at every number. So the limit as x approaches 1 is the same as the value of $f(x)$ at $x = 1$.

$$\begin{aligned} \lim_{x \to 1} (x^4 - 2x^2 + x + 3) &= 1^4 - 2 \cdot 1^2 + 1 + 3 && \textit{Replace x with 1.} \\ &= 1 - 2 + 1 + 3 \\ &= 3 \end{aligned}$$

The limit of $x^4 - 2x^2 + x + 3$ as x approaches 1 is 3.

b. $\lim_{x \to \pi} \frac{1 - \cos x}{x}$

Since the denominator of $\frac{1 - \cos x}{x}$ is not 0 at $x = \pi$, the function is continuous at $x = \pi$.

$$\begin{aligned} \lim_{x \to \pi} \frac{1 - \cos x}{x} &= \frac{1 - \cos \pi}{\pi} && \textit{Replace x with } \pi. \\ &= \frac{1 - (-1)}{\pi} \text{ or } \frac{2}{\pi} && \cos \pi = -1 \end{aligned}$$

The limit of $\frac{1 - \cos x}{x}$ as x approaches π is $\frac{2}{\pi}$.

Example 2 **Evaluate** $\lim_{x \to 4} \frac{x^2 - 9x + 20}{x - 4}$.

$$\begin{aligned} \lim_{x \to 4} \frac{x^2 - 9x + 20}{x - 4} &= \lim_{x \to 4} \frac{(x - 5)(x - 4)}{(x - 4)} && \textit{Factor.} \\ &= \lim_{x \to 4} (x - 5) \\ &= 4 - 5 && \textit{Replace x with 4.} \\ &= -1 \end{aligned}$$

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15-2

Study Guide

Derivatives and Antiderivatives

The derivative of a function $f(x)$ is another function, $f'(x)$, that gives the slope of the tangent line to $y = f(x)$ at any point. The following are some of the rules used to find the derivatives of polynomials.

Constant Rule:	The derivative of a constant function is zero. If $f(x) = c$, then $f'(x) = 0$.
Power Rule:	If $f(x) = x^n$, where n is a rational number, then $f'(x) = nx^{n-1}$.
Constant Multiple of a Power Rule:	If $f(x) = cx^n$, where c is a constant and n is a rational number, then $f'(x) = cnx^{n-1}$.

Finding the **antiderivative** $F(x)$ of a function $f(x)$ is the inverse of finding the derivative. The following are some of the rules used to find the antiderivative of a function.

Constant Multiple of a Power Rule:	If $f(x) = kx^n$, where n is a rational number other than -1 and k is a constant, the antiderivative is $F(x) = k \cdot \frac{1}{n+1}x^{n+1} + C$.
Sum and Difference Rule:	If the antiderivatives of $f(x)$ and $g(x)$ are $F(x)$ and $G(x)$, respectively, then the antiderivative of $f(x) \pm g(x)$ is $F(x) \pm G(x)$.

Example 1 **Find the derivative of each function.**

a. $\boldsymbol{f(x) = x^5}$

$$\begin{aligned} f'(x) &= 5x^{5-1} && \textit{Power Rule} \\ &= 5x^4 \end{aligned}$$

b. $\boldsymbol{f(x) = 3x + 2}$

$$\begin{aligned} f(x) &= 3x + 2 \\ &= 3x^1 + 2 && \textit{Rewrite } x \textit{ as a power.} \\ f'(x) &= 3 \cdot 1x^{1-1} + 0 && \textit{Constant Multiple of a Power Rule, Constant Rule, and Sum Rule} \\ &= 3x^0 \\ &= 3 && x^0 = 1 \end{aligned}$$

Example 2 **Find the antiderivative of $\boldsymbol{f(x) = 6x^2 - 10x + 4}$.**

$$\begin{aligned} f(x) &= 6x^2 - 10x + 4 \\ &= 6x^2 - 10x^1 + 4x^0 && \textit{Rewrite the function so that each term has a power of } x. \end{aligned}$$

Use the Constant Multiple of a Power and Sum and Difference Rules.

$$\begin{aligned} F(x) &= 6 \cdot \frac{1}{2+1}x^{2+1} + C_1 - \left(10 \cdot \frac{1}{1+1}x^{1+1} + C_2\right) + 4 \cdot \frac{1}{0+1}x^{0+1} + C_3 \\ &= \frac{6}{3}x^3 - \frac{10}{2}x^2 + \frac{4}{1}x^1 + C && \textit{Let } C = C_1 - C_2 + C_3. \\ &= 2x^3 - 5x^2 + 4x + C \end{aligned}$$

15-3

Study Guide

Area Under a Curve

Example **Use limits to find the area of the region between the graph of $y = x^2$ and the x-axis from $x = 0$ to $x = 5$. That is, find $\int_0^5 x^2\,dx$.**

Since $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, we must first find Δx and x_i.

$\Delta x = \frac{5-0}{n}$ or $\frac{5}{n}$ $\quad \Delta x = \frac{b-a}{n}$

$x_i = 0 + i \cdot \frac{5}{n}$ or $\frac{5i}{n}$ $\quad x_i = a + i\Delta x$

Now calculate the integral that gives the area.

$$\int_0^5 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i)^2 \Delta x \qquad f(x_i) = x_i^2$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5i}{n}\right)^2 \cdot \frac{5}{n} \qquad x_i = \frac{5i}{n},\ \Delta x = \frac{5}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{125i^2}{n^3} \qquad \textit{Multiply.}$$

$$= \lim_{n\to\infty} \left(\frac{125 \cdot 1^2}{n^3} + \frac{125 \cdot 2^2}{n^3} + \cdots + \frac{125 \cdot n^2}{n^3}\right)$$

$$= \lim_{n\to\infty} \frac{125}{n^3} \cdot (1^2 + 2^2 + \cdots + n^2) \qquad \textit{Factor.}$$

$$= \lim_{n\to\infty} \frac{125}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \qquad 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{250n^2 + 375n + 125}{6n^2} \qquad \textit{Multiply.}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(250 + \frac{375}{n} + \frac{125}{n^2}\right) \qquad \textit{Factor and divide by } n^2.$$

$$= \left(\lim_{n\to\infty} \frac{1}{6}\right)\left[\lim_{n\to\infty} 250 + \left(\lim_{n\to\infty} 375\right)\left(\lim_{n\to\infty} \frac{1}{n}\right) + \left(\lim_{n\to\infty} 125\right)\left(\lim_{n\to\infty} \frac{1}{n^2}\right)\right]$$

Limit theorems

$$= \frac{1}{6}[250 + (375)(0) + (125)(0)] \qquad \lim_{n\to\infty} \frac{1}{n} = 0,\ \lim_{n\to\infty} \frac{1}{n^2} = 0$$

$$= \frac{1}{6} \cdot 250$$

$$= \frac{250}{6} \text{ or } \frac{125}{3}$$

The area of the region is $\frac{125}{3}$ square units.

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15-4

Study Guide

The Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus** states that if $F(x)$ is the antiderivative of the continuous function $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

The statement may also be written $\int_a^b f(x)\,dx = F(x)\Big|_a^b$.

The antiderivative rules for **indefinite integrals,** denoted $\int f(x)\,dx$, are as follows.

Power Rule:	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, where n is a rational number and $n \neq -1$.
Constant Multiple of a Power Rule:	$\int kx^n\,dx = k \cdot \frac{1}{n+1}x^{n+1} + C$, where k is a constant, n is a rational number, and $n \neq -1$.
Sum and Difference Rule:	$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Example 1 **Evaluate $\int_2^5 x^2\,dx$.**

The antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3 + C$

$$\int_2^5 x^2\,dx = \frac{1}{3}x^3 + C\Big|_2^5 \qquad \textit{Fundamental Theorem of Calculus}$$

$$= \left(\frac{1}{3} \cdot 5^3 + C\right) - \left(\frac{1}{3} \cdot 2^3 + C\right) \qquad \textit{Let } x = 5 \textit{ and } 2 \textit{ and subtract.}$$

$$= \frac{117}{3} \text{ or } 39$$

Notice that the constant term C is eliminated when evaluating a definite integral using the Fundamental Theorem of Calculus.

Example 2 **Evaluate the indefinite integral $\int (3x^2 + 4x - 1)\,dx$.**

$$\int (3x^2 + 4x - 1)\,dx = \int (3x^2 + 4x^1 - x^0)\,dx \qquad x = x^1,\ 1 = x^0$$

$$= 3 \cdot \frac{1}{2+1}x^{2+1} + 4 \cdot \frac{1}{1+1}x^{1+1} - \left(\frac{1}{0+1}x^{0+1}\right) + C$$

Use antiderivative rules.

$$= \frac{3}{3}x^3 + \frac{4}{2}x^2 - \frac{1}{1}x^1 + C$$

$$= x^3 + 2x^2 - x + C \qquad \textit{Simplify.}$$